CONVOLUTIONS AND MOLLIFICATIONS

Henceforth, we use m to denote the Lebesgue measure on $\mathbb{R}$ and L^p to denote the Banach space $L^p(\mathbb{R},m)$ for any $p\in[1,\infty]$. One of the goals of this note is to prove that smooth functions with compact support are dense in L^p for any $p\in[1,\infty)$. In order to do so, we will introduce the notion of *convolution*, which is an important tool in analysis.

Convolutions and Young's Inequality. Let us start with a simple technical lemma.

Lemma 1. *Let $f,g\colon\mathbb{R}\to\mathbb{R}$ be measurable. Then, the functions*

$$x\mapsto\int\big(f(x-y)\cdot g(y)\big)^+\,\mathrm{d}y,\qquad \text{and}\qquad x\mapsto\int\big(f(x-y)\cdot g(y)\big)^-\,\mathrm{d}y$$

are well-defined and measurable.

Proof. The functions $(x,y)\mapsto x-y$ and $(x,y)\mapsto y$ are continuous from $\mathbb{R}^2$ to $\mathbb{R}$, hence their compositions $(x,y)\mapsto f(x-y)$ and $(x,y)\mapsto g(y)$ with f and g respectively are measurable. Thus, $(x,y)\mapsto f(x-y)g(y)$, as well as its positive and negative parts, are measurable too. An application of the Tonelli Theorem implies the claim. □

In light of the previous lemma, given two measurable functions f and g, we define their *convolution* to be

$$(f*g)(x)=\int f(x-y)\cdot g(y)\,\mathrm{d}y,$$

whenever the integral exists.

Exercise 2. *Using the definition, verify that $(f*g)(x)=(g*f)(x)$.*

For example, if $g(x)=\frac{1}{2r}\chi_{[-r,r]}(y)$ for some $r>0$, then the convolution $f*g$ is the average

$$(f*g)(x)=\frac{1}{2r}\int f(y)\cdot\chi_{[-r,r]}(y)\,\mathrm{d}y=\frac{1}{2r}\int_{x-r}^{x+r}f(y)\,\mathrm{d}y.$$

More in general, if g is a non-negative integrable function, we think of the convolution $(f*g)(x)$ as the "average" of f with respect to the measure $\mathrm{d}\mu=g(-y)\,\mathrm{d}y$ translated by x.

The following result establishes the integrability properties of the convolution of two functions.

Theorem 3 (Young's Inequality)**.** *Let $p,q,r\in[1,\infty]$ be such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$. For any $f\in L^p$ and $g\in L^q$, we have $f*g\in L^r$ and*

$$\|f*g\|_r\le\|f\|_p\cdot\|g\|_q.$$

To prove Young's Inequality we need the following generalization of Hölder's Inequality.

Proposition 4 (Hölder's Inequality for 3 functions)**.** *Let f_1,f_2,f_3 be measurable functions and let $a,b,c\in[1,\infty]$ be such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Then*

$$\int|f_1\cdot f_2\cdot f_3|\,\mathrm{d}m\le\|f_1\|_a\cdot\|f_2\|_b\cdot\|f_3\|_c.$$

Proof. If $a=\infty$, then

$$\int|f_1\cdot f_2\cdot f_3|\,\mathrm{d}m\le\|f_1\|_\infty\cdot\int|f_2\cdot f_3|\,\mathrm{d}m\le\|f_1\|_\infty\cdot\|f_2\|_b\cdot\|f_3\|_c,$$

where in the last step we used Hölder's Inequality, since $\frac{1}{b}+\frac{1}{c}=\frac{1}{\infty}+\frac{1}{b}+\frac{1}{c}=1$.

Otherwise, let $a' \in (1,\infty)$ be the conjugate exponent to a, namely $a' = \frac{a}{a-1}$, and define $\ell = \frac{b}{a'}$ and $\ell' = \frac{c}{a'}$. Notice that ℓ and ℓ' are conjugate exponents, since

$$\frac{1}{\ell} + \frac{1}{\ell'} = a'\left(\frac{1}{b} + \frac{1}{c}\right) = a'\left(1 - \frac{1}{a}\right) = 1.$$

Then, using Hölder's Inequality twice we get

$$\begin{aligned}\int |f_1 \cdot f_2 \cdot f_3| \,\mathrm{d}m &\le \|f_1\|_a \cdot \left(\int |f_2 \cdot f_3|^{a'} \,\mathrm{d}m\right)^{1/a'} \le \|f_1\|_a \cdot \|f_2\|_{a'\ell} \cdot \|f_3\|_{a'\ell'} \\ &= \|f_1\|_a \cdot \|f_2\|_b \cdot \|f_3\|_c,\end{aligned}$$

which completes the proof. □

Exercise 5. *Prove the following* Generalized Hölder's Inequality*: for any $n \ge 2$, any $p_1, \dots, p_n \in [1,\infty]$ such that $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1$, and for all measurable functions $f_1, \dots, f_n$, we have*

$$\|f_1 \cdots f_n\|_1 \le \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

We are now ready to prove Young's Inequality.

Proof of Young's Inequality. If $r = \infty$, then, for every $x \in \mathbb{R}$, Hölder's Inequality yields

$$|(f*g)(x)| \le \int |f(x-y)| \cdot |g(y)| \,\mathrm{d}y \le \left(\int |f(x-y)|^p \,\mathrm{d}y\right)^{1/p} \cdot \left(\int |g(y)|^q \,\mathrm{d}y\right)^{1/q} = \|f\|_p \cdot \|g\|_q,$$

which implies the claim.

Consider now the case $r < \infty$. By definition, we have

$$\begin{aligned}|(f*g)(x)| &\le \int |f(x-y)| \cdot |g(y)| \,\mathrm{d}y \\ &= \int (|f(x-y)|^p \cdot |g(y)|^q)^{1/r} \cdot |f(x-y)|^{1-p/r} \cdot |g(y)|^{1-q/r} \,\mathrm{d}y.\end{aligned}$$

Since

$$\frac{1}{r} + \frac{r-p}{pr} + \frac{r-q}{qr} = \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1,$$

we use the Hölder's Inequality for 3 functions with $a = r$, $b = \frac{pr}{r-p}$ and $c = \frac{qr}{r-q}$ and we obtain

$$\begin{aligned}&|(f*g)(x)| \\ &\le \left(\int |f(x-y)|^p \cdot |g(y)|^q \,\mathrm{d}y\right)^{1/r} \cdot \left(\int |f(x-y)|^{(1-\frac{p}{r})\frac{pr}{r-p}} \,\mathrm{d}y\right)^{\frac{r-p}{pr}} \cdot \left(\int |g(y)|^{(1-\frac{q}{r})\frac{qr}{r-q}} \,\mathrm{d}y\right)^{\frac{r-q}{qr}} \\ &= \left(\int |f(x-y)|^p \cdot |g(y)|^q \,\mathrm{d}y\right)^{1/r} \cdot \left(\int |f(x-y)|^p \,\mathrm{d}y\right)^{\frac{1}{p}\cdot\frac{r-p}{r}} \cdot \left(\int |g(y)|^q \,\mathrm{d}y\right)^{\frac{1}{q}\cdot\frac{r-q}{r}} \\ &= \left(\int |f(x-y)|^p \cdot |g(y)|^q \,\mathrm{d}y\right)^{1/r} \cdot \|f\|_p^{\frac{r-p}{r}} \cdot \|g\|_q^{\frac{r-q}{r}}.\end{aligned}$$

The inequality above holds also in the case $p = 1$ and $q = r$ (or if $q = 1$ and $p = r$), but it suffices to use the classical Hölder inequality to prove it.

Integrating, we obtain

$$\|f*g\|_r^r = \int |(f*g)(x)|^r \,\mathrm{d}x \le \|f\|_p^{r-p} \cdot \|g\|_q^{r-q} \int \left(\int |f(x-y)|^p \cdot |g(y)|^q \,\mathrm{d}y\right) \mathrm{d}x.$$

By Tonelli's Theorem, we can swap the order of integration and conclude

$$\begin{aligned}\|f*g\|_r^r &\leq \|f\|_p^{r-p}\cdot\|g\|_q^{r-q}\int\int |f(x-y)|^p\cdot|g(y)|^q\,\mathrm{d}x\,\mathrm{d}y\\ &= \|f\|_p^{r-p}\cdot\|g\|_q^{r-q}\int\left(\int|f(x-y)|^p\,\mathrm{d}x\right)|g(y)|^q\,\mathrm{d}y = \|f\|_p^r\cdot\|g\|_q^r,\end{aligned}$$

which proves the theorem. □

Mollifiers. We now introduce the concept of *mollification* of a function. Let us fix a non-negative, infinitely differentiable function $\varphi\colon \mathbb{R}\to\mathbb{R}_{\geq 0}$ with compact support. Here, we can choose once and for all

(1)
$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x|<1,\\ 0 & \text{otherwise.}\end{cases}$$

Exercise 6. *Verify that the function φ defined in* (1) *belongs to $\mathscr{C}_c^\infty(\mathbb{R})$, the space of infinitely differentiable functions with compact support.*

Up to multiplying φ by a positive scalar, we can assume that $\int\varphi\,\mathrm{d}m = 1$. For any $\varepsilon>0$, we define

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right).$$

Notice that φ_ε is supported in $[-\varepsilon,\varepsilon]$ and still has integral 1.

Theorem 7. *Let $p\in[1,\infty]$ and $f\in L^p$.*

(a) *For any $\varepsilon>0$, the function $f*\varphi_\varepsilon$ is infinitely differentiable and*

$$\frac{\mathrm{d}^n}{(\mathrm{d}x)^n}(f*\varphi_\varepsilon) = f*\left(\frac{\mathrm{d}^n}{(\mathrm{d}x)^n}\varphi_\varepsilon\right).$$

(b) *We have $f*\varphi_\varepsilon\to f$ almost everywhere as $\varepsilon\to 0$.*

(c) *If $p<\infty$, then $f*\varphi_\varepsilon\to f$ in L^p as $\varepsilon\to 0$.*

For any $\varepsilon>0$, the smooth function $f*\varphi_\varepsilon$ is called a *mollification* of f, and φ_ε is called a *mollifier*.

Proof. Let us prove (a) for $n=1$, the general case follows by induction and is left as an exercise to the reader. Fix $\varepsilon>0$ and let $h\neq 0$. By the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\frac{(f*\varphi_\varepsilon)(x+h)-(f*\varphi_\varepsilon)(x)}{h} &= \frac{1}{h}\int f(y)\cdot(\varphi_\varepsilon(x+h-y)-\varphi_\varepsilon(x-y))\,\mathrm{d}y\\ &= \int f(y)\cdot\left(\frac{1}{h}\int_0^h\varphi_\varepsilon'(x-y+\xi)\,\mathrm{d}\xi\right)\mathrm{d}y.\end{aligned}$$

For any fixed $x\in\mathbb{R}$ and $|h|\leq 1$, the function in brackets above (seen as a function of y) is supported inside a bounded interval I_x centered at x of diameter independent of h. Thus, the integrand function above is bounded by

$$\left|f(y)\cdot\left(\frac{1}{h}\int_0^h\varphi_\varepsilon'(x-y+\xi)\,\mathrm{d}\xi\right)\right| \leq \|\varphi'\|_\infty\cdot|f(y)|\cdot\chi_{I_x}(y),$$

which is integrable, since $f\in L^p$. By the Dominated Convergence Theorem,

$$\begin{aligned}\lim_{h\to 0}\frac{(f*\varphi_\varepsilon)(x+h)-(f*\varphi_\varepsilon)(x)}{h} &= \int f(y)\cdot\lim_{h\to 0}\left(\frac{1}{h}\int_0^h\varphi_\varepsilon'(x-y+\xi)\,\mathrm{d}\xi\right)\mathrm{d}y\\ &= \int f(y)\cdot\varphi_\varepsilon'(x-y)\,\mathrm{d}y = (f*\varphi_\varepsilon')(x).\end{aligned}$$

Let us show (b). For any fixed $x \in \mathbb{R}$, we have

$$|(f * \varphi_\varepsilon)(x) - f(x)| = \left| \int (f(y) - f(x)) \cdot \varphi_\varepsilon(x-y)\,\mathrm{d}y \right| \le \frac{1}{\varepsilon} \int |f(y) - f(x)| \cdot \varphi\left(\frac{x-y}{\varepsilon}\right) \mathrm{d}y.$$

Since the function $y \mapsto \varphi\left(\frac{x-y}{\varepsilon}\right)$ is supported in the interval $I_\varepsilon = [x-\varepsilon, x+\varepsilon]$, we obtain

$$|(f * \varphi_\varepsilon)(x) - f(x)| \le \frac{2\|\varphi\|_\infty}{|I_\varepsilon|} \int_{I_\varepsilon} |f(y) - f(x)|\,\mathrm{d}y.$$

By the Lebesgue Differentiation Theorem, the last term tends to 0 as $\varepsilon \to 0$ for almost every $x \in \mathbb{R}$.

Let us finish the proof by showing (c). Fix $\delta > 0$ and let us prove that there exists $\overline{\varepsilon} > 0$ such that $\|f * \varphi_\varepsilon - f\|_p \le \delta$ for all $\varepsilon \le \overline{\varepsilon}$.

By the density of $\mathscr{C}_c(\mathbb{R})$ in L^p, there exists a continuous function g, with support inside a bounded interval $[-K, K]$ for some $K > 0$, such that $\|f - g\|_p \le \delta/3$. By (b), we have that $g * \varphi_\varepsilon \to g$ as $\varepsilon \to 0$ almost everywhere. Since $g * \varphi_\varepsilon$ is a continuous function with support contained in $[-K-1, K+1]$ for all $\varepsilon \le 1$, the Dominated Convergence Theorem implies that $g * \varphi_\varepsilon \to g$ in L^p as $\varepsilon \to 0$. Hence, there exists $\overline{\varepsilon} > 0$ so that $\|g * \varphi_\varepsilon - g\|_p \le \delta/3$ for all $\varepsilon \le \overline{\varepsilon}$. Thus,

$$\|f - f * \varphi_\varepsilon\|_p \le \|f - g\|_p + \|g - g * \varphi_\varepsilon\|_p + \|g * \varphi_\varepsilon - f * \varphi_\varepsilon\|_p \le 2\delta/3 + \|(g - f) * \varphi_\varepsilon\|_p.$$

By Young's Inequality with $p = r$ and $q = 1$, and since $\|\varphi_\varepsilon\|_1 = \int \varphi_\varepsilon \,\mathrm{d}m = 1$,

$$\|f - f * \varphi_\varepsilon\|_p \le 2\delta/3 + \|g - f\|_p \cdot \|\varphi_\varepsilon\|_1 \le \delta.$$

The proof is therefore complete. □

As a consequence of the previous theorem, we deduce the following density result.

Corollary 8. *The space* $\mathscr{C}_c^\infty(\mathbb{R})$ *is dense in* L^p *for all* $p \in [1, \infty)$.

Exercise 9. *Prove Corollary 8.*