MEASURE AND INTEGRATION THEORY—VO 250076
MOCK EXAM

The duration of the exam is **2 hours and 30 minutes**.

The exam consists of **5 exercises**.

The total score is **100 points**. The minimum score to pass the exam is **50 points**. The marks will be assigned using the following conversion:

50-59 points $\longrightarrow$ mark: 4.

60-69 points $\longrightarrow$ mark: 3.

70-84 points $\longrightarrow$ mark: 2.

85-100 points $\longrightarrow$ mark: 1.

Please turn the page only when instructed to do so.

Exercise 1. *Let $(X, \mathscr{A}, \mu)$ be a measure space and let $f_n, f\colon X \to \mathbb{R}$, for $n \in \mathbb{N}$, be functions. Prove or disprove the following statements.*

(1) *If $f_n \to f$ pointwise and if the functions f_n are measurable for all $n \in \mathbb{N}$, then f is measurable.*
[5 points]

(2) *If $f_n \to f$ pointwise and if the functions f_n are integrable for all $n \in \mathbb{N}$, then f is integrable.*
[5 points]

(3) *If $f_n \to f$ pointwise and if the functions f_n are measurable and there exists $C \geq 0$ such that $\int |f_n| \,\mathrm{d}\mu \leq C$ for all $n \in \mathbb{N}$, then f is integrable and $\int |f| \,\mathrm{d}\mu \leq C$.*
[6 points]

Exercise 2. *Let $(X, \mathscr{A}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $f\colon X \to \mathbb{R}$ be a measurable function.*

(1) *Prove that $x \mapsto |\cos(f(x))|^n$ is a measurable function for all $n \in \mathbb{N}$ and prove that the set $A = \{x \in X \;:\; f(x) \in \pi\mathbb{Z}\}$ is measurable.*
[8 points]

(2) *Calculate, justifying all steps, the limit*

$$\lim_{n\to\infty} \int |\cos(f(x))|^n \,\mathrm{d}\mu.$$

[8 points]

(3) *Show that, in general, the result in the previous part is false if $\mu(X) = \infty$.*
[4 points]

Exercise 3. *Let $(X, \mathscr{A}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $f_n, f\colon X \to \mathbb{R}$, for $n \in \mathbb{N}$, be measurable functions.*

(1) *Give the precise definitions of the following:*
- *f_n converges to f μ-almost everywhere.*
- *f_n converges to f in measure.*
- *f_n converges to f in L^2.*

[6 points]

(2) *Prove that f_n converges to f μ-almost everywhere if and only if*

$$\mu\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} \{x \in X \;:\; |f_k(x) - f(x)| \geq \varepsilon\}\right) = 0 \quad \textit{for every } \varepsilon > 0.$$

[6 points]

(3) *Prove that f_n converges to f μ-almost everywhere if and only if the sequence of functions*

$$g_n = \sup_{k\geq n} |f_k - f|$$

converges to 0 in measure.
[6 points]

(4) *Show that, in general, the result in the previous part is false if $\mu(X) = \infty$.*
[6 points]

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Exercise 4. *Let m_2 denote the Lebesgue measure on $Q = [0,1]^2$ (with the Borel σ-algebra).*

(1) *For any $p \in [1,\infty]$, define the p-norm $\|\cdot\|_p$ and the spaces $\mathcal{L}^p(Q,m_2)$ and $L^p(Q,m_2)$.*
[10 points]

(2) *Consider the function $f\colon Q \to \mathbb{R}$ defined by*

$$f(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{1}{\sqrt{|x-y|}} & \text{if } x \neq y. \end{cases}$$

Determine for which $p \in [1,\infty]$ the function f belongs to $\mathcal{L}^p(Q,m_2)$ and, for those p, calculate $\|f\|_p$.
[10 points]

Exercise 5. *Let μ be a signed measure on the measurable space $(X,\mathscr{A})$.*

(1) *Define what it means for $E \in \mathscr{A}$ to be a negative set, a positive set, and a null-set for μ.*
[4 points]

(2) *State the Hahn Decomposition Theorem and the Jordan Decomposition Theorem.*
[10 points]

(3) *Give two equivalent definitions of the total variation of μ.*
[6 points]