Ergodic and mixing properties of horocycle flows and their time-changes

Simons Semester – Lecture 1

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Plan of the course

1. The setting: $SL_2(\mathbb{R})$, the horocycle flow, and many important objects.
2. Ergodicity: quantitative results on horocycle ergodic averages.
3. Mixing: the "mixing via shearing" method and the decay of correlations.
4. Time-changes: definition and basic facts; ergodicity and mixing.
5. Rigidity: Ratner's rigidity result.
6. Beyond $SL_2(\mathbb{R})$: time-changes and other perturbations in other settings.

Plan for today

In the first part, we will introduce

- the group $G = \mathrm{SL}_2(\mathbb{R})$ and its Lie algebra,
- the homogeneous flows on its quotients,
- the Haar measure.

In the second part, we will define

- the Adjoint action and the Lie brackets,
- the Killing form and the action of G on the hyperbolic plane,
- the Casimir operator.

Homogeneous flows on $SL_2(\mathbb{R})$

The group $\mathrm{SL}_2(\mathbb{R})$

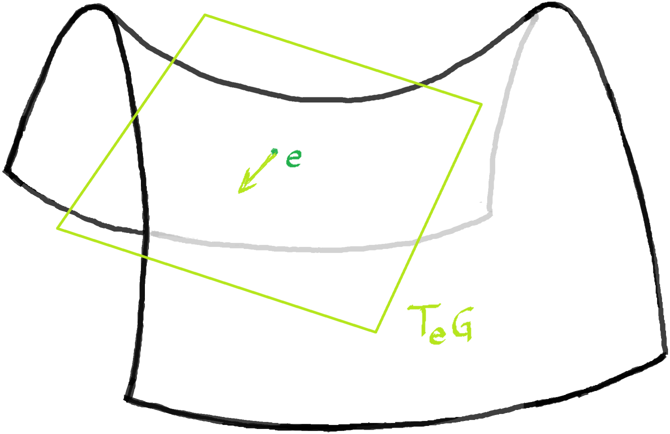
$$G := \mathrm{SL}_2(\mathbb{R}) = \left\{ \mathbf{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) : \right.$$

$$\left. a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}$$

We think of G as an embedded submanifold of $\mathbb{R}^4$.

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm \mathbf{e}\}, \qquad \text{and} \qquad \mathbf{e} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The group $SL_2(\mathbb{R})$



Tangent spaces

$$\mathfrak{g} := T_{\mathbf{e}}G = \Big\{ \gamma'(0) \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) : \gamma\colon (-\varepsilon,\varepsilon) \to G \text{ is a smooth curve and } \gamma(0) = \mathbf{e} \Big\}$$

For example,

$$\gamma_1(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \mathbf{u} := {\gamma_1}'(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g},$$

$$\gamma_2(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \quad \Rightarrow \quad \mathbf{x} := {\gamma_2}'(0) = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \in \mathfrak{g},$$

$$\gamma_3(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v} := {\gamma_3}'(0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}.$$

Tangent spaces

Lemma

$$\mathfrak{g} = \{\mathbf{g} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \ : \ \mathrm{tr}(\mathbf{g}) = 0\}$$

Proof (sketch).

- $\mathfrak{g}$ is a vector space: let γ, η be such that $\gamma'(0), \eta'(0) \in \mathfrak{g}$ and let $a \in \mathbb{R}$; then

$$(\gamma \cdot \eta)'(0) = \gamma'(0) \cdot \eta(0) + \gamma(0) \cdot \eta'(0) = \gamma'(0) + \eta'(0)$$
$$(\gamma_a)'(0) = a \cdot \gamma'(0), \qquad \text{where} \qquad \gamma_a(t) = \gamma(at).$$

- $\langle \mathbf{u}, \mathbf{x}, \mathbf{v} \rangle = \{\mathrm{tr}(\mathbf{g}) = 0\} \subseteq \mathfrak{g}$.
- if $t \mapsto \gamma(t) \in G$ is a smooth curve with $\gamma(0) = \mathbf{e}$,

$$1 = \det(\gamma(t)) \quad \Rightarrow \quad 0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det(\gamma(t)) = \mathrm{tr}(\gamma'(0)).$$

Tangent spaces

More in general, for any $\mathbf{g} \in G$,

$$T_{\mathbf{g}}G = \mathbf{g} \cdot \mathfrak{g} = \{\mathbf{g}\,\mathbf{w} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \ : \ \mathbf{w} \in \mathfrak{g}\} \simeq \mathfrak{g}.$$

Any $\mathbf{w} \in \mathfrak{g}$ can be identified with a *smooth* vector field W, where

$$W(\mathbf{g}) = \mathbf{g}\,\mathbf{w} \in T_{\mathbf{g}}G.$$

Matrix exponential

For any $\mathbf{w} \in \mathfrak{g}$, define

$$\exp(\mathbf{w}) = \sum_{k=0}^{\infty} \frac{\mathbf{w}^k}{k!} = \mathbf{e} + \mathbf{w} + \frac{1}{2}\mathbf{w}^2 + \cdots .$$

Since $\det(\exp(\mathbf{w})) = e^{\mathrm{tr}(\mathbf{w})}$, we have $\exp(\mathfrak{g}) \subseteq G$.

$$t\,\mathbf{u} = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \exp(t\,\mathbf{u}) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

$$t\,\mathbf{x} = \begin{pmatrix} t/2 & 0 \\ 0 & -t/2 \end{pmatrix} \quad \Rightarrow \quad \exp(t\,\mathbf{x}) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

$$t\,\mathbf{r} = \begin{pmatrix} 0 & -t/2 \\ t/2 & 0 \end{pmatrix} \quad \Rightarrow \quad \exp(t\,\mathbf{r}) = \begin{pmatrix} \cos(t/2) & -\sin(t/2) \\ \sin(t/2) & \cos(t/2) \end{pmatrix}.$$

Matrix exponential

For $\mathbf{w} \in \mathfrak{g}$ we define the flow $(\varphi_t^{\mathbf{w}})_{t\in\mathbb{R}}$ on G by

$$\varphi_t^{\mathbf{w}}(\mathbf{g}) = \mathbf{g} \exp(t\,\mathbf{w}).$$

The flow $(\varphi_t^{\mathbf{w}})_{t\in\mathbb{R}}$ is generated by the vector field W:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \varphi_t^{\mathbf{w}}(\mathbf{g}) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathbf{g} \exp(t\,\mathbf{w}) = \mathbf{g}\,\mathbf{w} = W(\mathbf{g}).$$

We identify W with the derivation

$$Wf(\mathbf{g}) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f \circ \varphi_t^{\mathbf{w}}(\mathbf{g}), \qquad \text{for } f \in \mathscr{C}^1(G).$$

Homogeneous flows on quotients

Lemma

For any discrete subgroup $\Gamma \leq G$, the quotient $M = \Gamma \backslash G$ is a smooth manifold.

Proof (sketch). Since Γ is discrete, it acts properly discontinuously on G and the projection $G \to M$ is a covering map Then, M inherits the smooth structure from G.

The flow $(\varphi_t^{\mathbf{w}})_{t\in\mathbb{R}}$ generated by $\mathbf{w} \in \mathfrak{g}$ is well-defined on M: for $\Gamma\mathbf{g} \in M$, we let

$$\varphi_t^{\mathbf{w}}(\Gamma\mathbf{g}) = \Gamma\mathbf{g}\exp(t\,\mathbf{w}).$$

We will be interested in quotients M which are *compact*.

The Haar measure

- Recall that, for any $\mathbf{g} \in G$, we can identify $T_{\mathbf{g}}G = \{\mathbf{g}\,\mathbf{w} : \mathbf{w} \in \mathfrak{g}\}$ with $\mathfrak{g}$.
- Fix the basis $\mathbf{u}, \mathbf{x}, \mathbf{v}$ of $\mathfrak{g}$ as above and identify all $T_{\mathbf{g}}G$ with $\mathbb{R}^3$.
- Given $\mathbf{g} \in G$, define $\omega_{\mathbf{g}}$ by

$$\omega_{\mathbf{g}}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ x_1 & x_2 & x_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

for any $\mathbf{w}_i = (u_i, x_i, v_i) \in T_{\mathbf{g}}G = \mathbb{R}^3$, $i = 1, 2, 3$.

The Haar measure

We define a measure vol on G by

$$\mathrm{vol}(f) = \int_G f\,|\omega|$$

for any continuous function $f\colon G \to \mathbb{R}$ with compact support.

Proposition

The measure vol satisfies the following properties:

- it is well-defined on any quotient $M = \Gamma\backslash G$ for any discrete subgroup $\Gamma \leq G$.
- it is invariant under $(\varphi_t^{\mathbf{w}})_{t\in\mathbb{R}}$ for any $\mathbf{w} \in \mathfrak{g}$.
- up to scalar multiples, it is the unique measure with these properties.

Lie brackets, the Killing form, and the Casimir operator

Adjoint and Lie brackets

Recall that for any two matrices $\mathbf{w}, \mathbf{z}$ we have $\mathrm{tr}(\mathbf{w}\mathbf{z}) = \mathrm{tr}(\mathbf{z}\mathbf{w})$.

Observations

- G acts on $\mathfrak{g}$ by conjugation: for any $\mathbf{g} \in G$, the map

$$\mathrm{Ad}(\mathbf{g}) \colon \mathbf{w} \mapsto \mathbf{g}^{-1}\mathbf{w}\mathbf{g}$$

  is a linear automorphism of $\mathfrak{g}$.
- $\mathfrak{g}$ is closed under the bracket operation defined by

$$[\mathbf{w}, \mathbf{z}] := \mathbf{w}\mathbf{z} - \mathbf{z}\mathbf{w}.$$

Adjoint and Lie brackets

If $\mathbf{g} = \exp(t\mathbf{w})$, then, under the identification $\mathfrak{g} \simeq T_{\mathbf{g}} G$,

$$\mathrm{Ad}(\exp(t\mathbf{w}))\colon \mathfrak{g} \to \mathfrak{g} \qquad \simeq \qquad D\varphi_t^{\mathbf{w}}\colon \mathfrak{g} \to T_{\mathbf{g}} G.$$

Moreover,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}(\exp(t\mathbf{w}))(\mathbf{z}) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(-t\mathbf{w})\,\mathbf{z}\,\exp(t\mathbf{w}) = -\mathbf{w}\,\mathbf{z} + \mathbf{z}\,\mathbf{w} \\ &= -[\mathbf{w}, \mathbf{z}],\end{aligned}$$

in other words

$$\mathfrak{ad}_{\mathbf{w}} := [\mathbf{w}, \cdot] = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} D\varphi_{-t}^{\mathbf{w}} = \mathscr{L}_W.$$

Adjoint and Lie brackets

Fix the basis

$$\mathbf{r} = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix},$$

and identify $\mathfrak{g}$ with $\mathbb{R}^3$.

We compute the 3×3 matrices representing the linear endomorphisms $\mathfrak{ad}_{\mathbf{r}}$, $\mathfrak{ad}_{\mathbf{x}}$, and $\mathfrak{ad}_{\mathbf{y}}$:

$$\mathfrak{ad}_{\mathbf{r}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \qquad \mathfrak{ad}_{\mathbf{x}} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}, \qquad \mathfrak{ad}_{\mathbf{y}} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Killing Form

Definition

The *Killing Form* is the bilinear form on $\mathfrak{g}$ defined by

$$\mathrm{B}(\mathbf{w}, \mathbf{z}) = 2\,\mathrm{tr}(\mathfrak{ad}_{\mathbf{w}} \circ \mathfrak{ad}_{\mathbf{z}}).$$

With respect to the basis $\{\mathbf{r}, \mathbf{x}, \mathbf{y}\}$, we can write

$$\mathrm{B}(\mathbf{w}, \mathbf{z}) = \mathbf{w}^t \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{z},$$

in particular it is *non-degenerate*.

The Killing Form

Recall that G acts on $\mathfrak{g}$ by conjugation: $\mathrm{Ad}(g)\colon \mathbf{w} \mapsto \mathbf{g}^{-1}\mathbf{w}\,\mathbf{g}$.

Lemma

The Killing Form is invariant by the action of G, namely for all $\mathbf{g} \in G$ and for all $\mathbf{w}, \mathbf{z} \in \mathfrak{g}$ we have

$$\mathrm{B}(\mathrm{Ad}(\mathbf{g})\mathbf{w}, \mathrm{Ad}(\mathbf{g})\mathbf{z}) = \mathrm{B}(\mathbf{w}, \mathbf{z}).$$

Proof (sketch).

- By direct computation, verify that

$$\mathfrak{ad}_{\mathbf{g}^{-1}\mathbf{w}\,\mathbf{g}} = \mathrm{Ad}(\mathbf{g}) \circ \mathfrak{ad}_{\mathbf{w}} \circ \mathrm{Ad}(\mathbf{g})^{-1}.$$

- By invariance of the trace under conjugation,

$$\begin{aligned} \mathrm{B}(\mathrm{Ad}(\mathbf{g})\mathbf{w}, \mathrm{Ad}(\mathbf{g})\mathbf{z}) &= 2\,\mathrm{tr}(\mathfrak{ad}_{\mathbf{g}^{-1}\mathbf{w}\,\mathbf{g}} \circ \mathfrak{ad}_{\mathbf{g}^{-1}\mathbf{z}\,\mathbf{g}}) \\ &= 2\,\mathrm{tr}(\mathrm{Ad}(\mathbf{g}) \circ \mathfrak{ad}_{\mathbf{w}} \circ \mathfrak{ad}_{\mathbf{z}} \circ \mathrm{Ad}(\mathbf{g})^{-1}) = 2\,\mathrm{tr}(\mathfrak{ad}_{\mathbf{w}} \circ \mathfrak{ad}_{\mathbf{z}}) = \mathrm{B}(\mathbf{w}, \mathbf{z}) \end{aligned}$$

The hyperbolic plane

The set

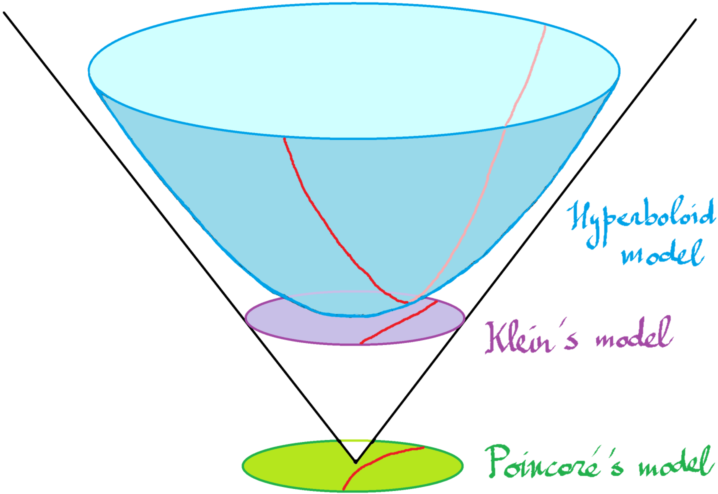
$$\begin{aligned}\mathscr{H} :=& \{\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3 : \mathrm{B}(\mathbf{z}, \mathbf{z}) = -1,\ z_1 > 0\} \\ =& \{\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3 : z_1^2 - z_2^2 - z_3^2 = 1,\ z_1 > 0\}.\end{aligned}$$

equipped with the distance

$$d_{\mathscr{H}}(\mathbf{w}, \mathbf{z}) = \operatorname{arcosh}(-\mathrm{B}(\mathbf{w}, \mathbf{z})) = \operatorname{arcosh}(w_1 z_1 - w_2 z_2 - w_3 z_3)$$

is the hyperboloid model of the hyperbolic plane.

The hyperbolic plane



Action of G on the hyperbolic plane

Proposition

- G and $\mathrm{PSL}_2(\mathbb{R}) = G/\{\pm \mathbf{e}\}$ act on $\mathscr{H}$ by isometries.
- The action is transitive.
- The stabilizer of $\mathbf{r} = (1,0,0)$ is $\mathrm{SO}(2)$.

Proof (sketch).

- Since G is connected, it acts on $\mathscr{H}$; moreover, the action of $-\mathbf{e}$ is trivial. Since $\mathrm{Ad}(\mathbf{g})$ preserves B, it is an isometry for the hyperbolic distance $d_{\mathscr{H}}$.
- Given $(r,x,y) \in \mathscr{H}$, let $\mathbf{g} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{r+y} & 0 \\ 0 & 1/\sqrt{r+y} \end{pmatrix}$. Then, $\mathrm{Ad}(\mathbf{g})\mathbf{r} = (r,x,y)$.
- Follows from the fact that $\mathrm{SO}(2) = \exp(\mathbb{R}\,\mathbf{r})$.

Action of G on the hyperbolic plane

We proved that $G/\mathrm{SO}(2)$ can be identified with $\mathscr{H}$.

One can prove that:

- $\mathrm{PSL}_2(\mathbb{R}) = G/\{\pm \mathbf{e}\}$ can be identified with the unit tangent bundle of $\mathscr{H}$.
- $M = \Gamma \backslash G$ can be identified with (possibly, a double cover of) the unit tangent bundle of $S = \Gamma \backslash \mathscr{H}$.
- The flow $(\varphi_t^{\mathbf{x}})_{t \in \mathbb{R}}$ on M is the *geodesic flow* and $(\varphi_t^{\mathbf{u}})_{t \in \mathbb{R}}$ is the *stable horocycle flow*.

The Casimir operator

Recall that

- the Killing form B is non degenerate. The *dual basis* of $\{\mathbf{r}, \mathbf{x}, \mathbf{y}\}$ with respect to B is $\{-\mathbf{r}, \mathbf{x}, \mathbf{y}\}$.
- the elements $\mathbf{r}, \mathbf{x}, \mathbf{y} \in \mathfrak{g}$ are identified with vector fields R, X, Y on G (and on its quotients).

The second order differential operator

$$\square = R^2 - X^2 - Y^2$$

is called the *Casimir operator*.

The Casimir operator

Proposition

- The definition of the Casimir operator does not depend of the choice of the basis $\{\mathbf{r}, \mathbf{x}, \mathbf{y}\}$.
- The Casimir operator commutes with any vector field W defined by $\mathbf{w} \in \mathfrak{g}$.
- On $\mathscr{C}^2$-functions f defined on $\mathscr{H} = G/\mathrm{SO}(2)$ (namely, for which $Rf = 0$), it coincides with minus the hyperbolic Laplacian.

Proof (sketch).

- Let $\{\mathbf{w}_i\}$ and $\{\mathbf{z}_j\}$ be two basis of $\mathfrak{g}$, and let $\{\widehat{\mathbf{w}_i}\}$ and $\{\widehat{\mathbf{z}}_j\}$ be their dual.
- If $\mathbf{w}_i = a_{ij}\mathbf{z}_j$, then $\widehat{\mathbf{z}}_i = a_{ji}\widehat{\mathbf{w}_j}$.
- Thus, $\mathbf{w}_i \widehat{\mathbf{w}_i} = a_{ij}\mathbf{z}_j\widehat{\mathbf{w}_i} = \mathbf{z}_j \widehat{\mathbf{z}}_j$.

The Casimir operator

Proof (sketch – cont'd).

- Need to prove that $\Box \mathbf{w} - \mathbf{w} \Box = 0$. Enough to consider $\mathbf{w} \in \{\mathbf{r}, \mathbf{x}, \mathbf{y}\}$.
- Direct computation in each case. E.g., for $\mathbf{w} = \mathbf{r}$,

$$\begin{aligned} \Box \mathbf{r} - \mathbf{r} \Box &= -\mathbf{x}^2 \mathbf{r} - \mathbf{y}^2 \mathbf{r} + \mathbf{r}\,\mathbf{x}^2 + \mathbf{r}\,\mathbf{y}^2 \\ &= -\mathbf{x}[\mathbf{x}, \mathbf{r}] - \mathbf{y}[\mathbf{y}, \mathbf{r}] + [\mathbf{r}, \mathbf{x}]\mathbf{x} + [\mathbf{r}, \mathbf{y}]\mathbf{y} \\ &= \frac{1}{2}\mathbf{x}\,\mathbf{y} - \frac{1}{2}\mathbf{y}\,\mathbf{x} + \frac{1}{2}\mathbf{y}\,\mathbf{x} - \frac{1}{2}\mathbf{x}\,\mathbf{y} = 0. \end{aligned}$$

The Casimir operator

Proof (sketch – cont'd).

- The hyperbolic metric on $\mathscr{H}$ is given by the restriction of the Killing form B.
- The vector fields X and Y on $\mathscr{H}$ are orthonormal.
- For any $\mathscr{C}^2$-function f,

$$\Delta_{\mathscr{H}} f = \operatorname{div}(\nabla f) = (X^2 + Y^2) f.$$

Summary

Summary

- For any discrete subgroup $\Gamma \leq G$, the quotient $M = \Gamma \backslash G$ is a smooth manifold.
- Any $\mathbf{w} \in \{\mathbf{g} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) : \mathrm{tr}(\mathbf{g}) = 0\}$ defines a smooth vector field on M and the flow $(\varphi_t^{\mathbf{w}})_{t\in\mathbb{R}}$ it generated is given by

$$\varphi_t^{\mathbf{w}} \colon \Gamma\mathbf{g} \mapsto \Gamma\mathbf{g}\exp(t\,\mathbf{w}).$$

- Any such flow preserves a smooth measure μ called the Haar measure.
- We computed $D\varphi_t^{\mathbf{w}} = \mathrm{Ad}(\exp(t\mathbf{w}))$ and $\mathscr{L}_W = \mathfrak{ad}_{\mathbf{w}}$.
- We introduced a second order differential operator $\Box$ which “extends” the hyperbolic Laplacian.