Ergodic and mixing properties of horocycle flows and their time-changes

Simons Semester – Lecture 2

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Plan for today

Today we will focus on the ergodic properties of the horocycle flows. We will study

- the asymptotics of ergodic averages of smooth functions,
- a temporal limit theorem.

Asymptotics of ergodic averages

The setting

- $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma \leq G$ is discrete,
- $M = \Gamma \backslash G$ is compact,
- vol is the Haar measure on M, normalized so that $\mathrm{vol}(M) = 1$.
- $\mathbf{u} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}$ generates the *horocycle flow*

$$h_t = \varphi_t^{\mathbf{u}} \colon \Gamma g \mapsto \Gamma g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

- $\mathbf{x} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \in \mathfrak{g}$ generates the *geodesic flow*

$$\varphi_t^{\mathbf{x}} \colon \Gamma g \mapsto \Gamma g \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}.$$

Unique ergodicity

Let $f \in \mathscr{C}(M)$. We want to study

$$A_f(p,t) := \frac{1}{t}\int_0^t f \circ h_r(p)\,\mathrm{d}r.$$

Theorem (Furstenberg 1973)

The horocycle flow on M is uniquely ergodic, namely

$$\lim_{t\to\infty}\left| A_f(p,t) - \int_M f\,\mathrm{d}\mathrm{vol}\right| = 0 \qquad \text{uniformly in } p \in M.$$

Proof. See Jon Chaika's course.

Can we say how fast $A_f(p,t) \to \int_M f\,\mathrm{d}\mathrm{vol}$?

The commutation relation

We compute

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} = \begin{pmatrix} e^{\frac{t}{2}} & re^{-\frac{t}{2}} \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \begin{pmatrix} 1 & e^{-t}r \\ 0 & 1 \end{pmatrix},$$

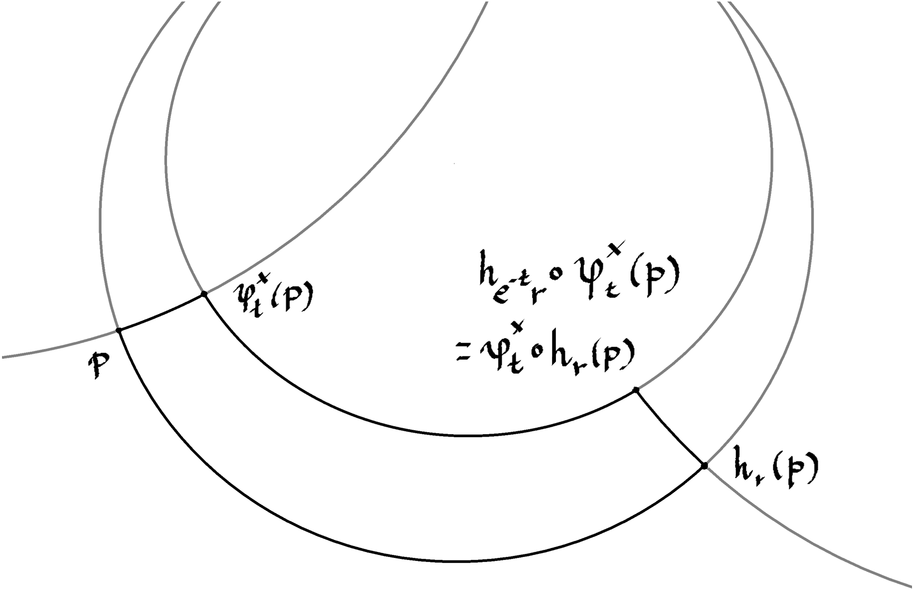
which means

$$\varphi_t^{\mathbf{x}} \circ h_r(p) = h_{e^{-t}r} \circ \varphi_t^{\mathbf{x}}(p) \qquad \text{for any } p \in M.$$

Equivalently,

- $D\varphi_t^{\mathbf{x}}(\mathbf{u}) \simeq \mathrm{Ad}(\exp(t\mathbf{x}))(\mathbf{u}) = e^{-t}\mathbf{u}$,
- $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} D\varphi_t^{\mathbf{x}}(\mathbf{u}) = -\mathscr{L}_X(U) = -[\mathbf{x}, \mathbf{u}] = -\mathbf{u}$.

The commutation relation



Ergodic averages

Let $f \in \mathscr{C}(M)$. We look at

$$A_f(p, e^t) := \frac{1}{e^t} \int_0^{e^t} f \circ h_r(p) \, \mathrm{d}r.$$

Since

$$h_r(p) = h_r \circ \varphi^{\mathbf{X}}_{-t} \circ \varphi^{\mathbf{X}}_t(p) = \varphi^{\mathbf{X}}_{-t} \circ h_{re^{-t}} \circ \varphi^{\mathbf{X}}_t(p),$$

we have

$$\begin{aligned} A_f(p, e^t) &= \frac{1}{e^t} \int_0^{e^t} f \circ h_r(p) \, \mathrm{d}r = \frac{1}{e^t} \int_0^{e^t} f \circ \varphi^{\mathbf{X}}_{-t} \circ h_{re^{-t}} \circ \varphi^{\mathbf{X}}_t(p) \, \mathrm{d}r \\ &= \int_0^1 f \circ \varphi^{\mathbf{X}}_{-t} \circ h_r\big(\varphi^{\mathbf{X}}_t(p)\big) \, \mathrm{d}r. \end{aligned}$$

Ergodic averages

We obtained

$$A_f(p, e^t) = \int_0^1 f \circ \varphi^{\mathbf{X}}_{-t} \circ h_r\big(\varphi^{\mathbf{X}}_t(p)\big)\, \mathrm{d}r.$$

Define

$$J_f(p, t) := \int_0^1 f \circ \varphi^{\mathbf{X}}_{-t} \circ h_r(p)\, \mathrm{d}r,$$

we proved the following fact.

Lemma

For any $f \in \mathscr{C}(M)$,

$$A_f(p, t) = J_f\big(\varphi^{\mathbf{X}}_{\log t}(p), \log t\big).$$

The function J

Assume now $f \in \mathscr{C}^2(M)$, so that

$$\Box f \in \mathscr{C}(M), \qquad \text{where} \qquad \Box = R^2 - X^2 - Y^2.$$

Fix $p \in M$; we study

$$J(t) = J_f(p,t) = \int_0^1 f \circ \varphi^{\mathbf{x}}_{-t} \circ h_r(p)\,\mathrm{d}r.$$

Proposition

Assume that $f \in \mathscr{C}^2(M)$ satisfies $\Box f = \mu f$ for some $\mu \in \mathbb{R}$. Then,

$$J''(t) + J'(t) + \mu J(t) = e^{-t}\big(Vf \circ \varphi^{\mathbf{x}}_{-t}(p) - Vf \circ \varphi^{\mathbf{x}}_{-t} \circ h_1(p)\big),$$

where $\mathbf{v} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}$.

Proof of the Proposition

Instead of the basis $\{\mathbf{r}, \mathbf{x}, \mathbf{y}\}$, we express $\Box$ in terms of the basis

$$\mathbf{u} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so that

$$\mathbf{r} = \frac{1}{2}(\mathbf{v} - \mathbf{u}), \qquad \text{and} \qquad \mathbf{y} = \frac{1}{2}(\mathbf{u} + \mathbf{v}).$$

Thus, since $\mathbf{vu} = \mathbf{uv} + [\mathbf{v}, \mathbf{u}] = \mathbf{uv} - 2\mathbf{x}$, we get

$$\begin{aligned} \Box = R^2 - X^2 - Y^2 &= \frac{1}{4}(V-U)(V-U) - X^2 - \frac{1}{4}(V+U)(V+U) \\ &= -X^2 - \frac{1}{2}(UV + VU) = -X^2 + X - UV. \end{aligned}$$

Proof of the Proposition

For $f \in \mathscr{C}^2(M)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \varphi^{\mathbf{X}}_{-t} = -Xf \circ \varphi^{\mathbf{X}}_{-t}, \qquad \text{and} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} f \circ \varphi^{\mathbf{X}}_{-t} = X^2 f \circ \varphi^{\mathbf{X}}_{-t}.$$

Recalling

$$J(t) = \int_0^1 f \circ \varphi^{\mathbf{X}}_{-t} \circ h_r(p)\,\mathrm{d}r, \qquad \text{and} \qquad \Box = -X^2 + X - UV,$$

we differentiate under the integral sign,

$$\int_0^1 (\Box f) \circ \varphi^{\mathbf{X}}_{-t} \circ h_r(p)\,\mathrm{d}r = -J''(t) - J'(t) - \int_0^1 (UVf) \circ \varphi^{\mathbf{X}}_{-t} \circ h_r(p)\,\mathrm{d}r.$$

Proof of the Proposition

If $\Box f = \mu f$, we deduce

$$J''(t) + J'(t) + \mu J(t) = -\int_0^1 (UVf) \circ \varphi^{\mathbf{x}}_{-t} \circ h_r(p)\,\mathrm{d}r.$$

Now we use the commutation $\varphi^{\mathbf{x}}_{-t} \circ h_r(p) = h_{e^t r} \circ \varphi^{\mathbf{x}}_{-t}(p)$ to get

$$\begin{aligned} -\int_0^1 (UVf) \circ \varphi^{\mathbf{x}}_{-t} \circ h_r(p)\,\mathrm{d}r &= -\int_0^1 (UVf) \circ h_{e^t r} \circ \varphi^{\mathbf{x}}_{-t}(p)\,\mathrm{d}r \\ &= -e^{-t}\int_0^{e^t} (UVf) \circ h_r \circ \varphi^{\mathbf{x}}_{-t}(p)\,\mathrm{d}r = -e^{-t}\Big[Vf \circ h_r \circ \varphi^{\mathbf{x}}_{-t}(p)\Big]_0^{e^t} \\ &= e^{-t}\Big(Vf \circ \varphi^{\mathbf{x}}_{-t}(p) - Vf \circ \varphi^{\mathbf{x}}_{-t} \circ h_1(p)\Big). \end{aligned}$$

Asymptotics for J

We need to solve

$$J''(t) + J'(t) + \mu J(t) = e^{-t} G(t),$$

where

$$G(t) = G_f(p,t) = Vf \circ \varphi^{\mathbf{X}}_{-t}(p) - Vf \circ \varphi^{\mathbf{X}}_{-t} \circ h_1(p).$$

Note that

$$\sup_{t \geq 0} \sup_{p \in M} |G_f(p,t)| \leq 2\|f\|_{\mathscr{C}^1}.$$

Let

$$\nu \in \mathbb{R}_{\geq 0} \cup \imath\mathbb{R}_{>0} \qquad \text{such that} \qquad \frac{1-\nu^2}{4} = \mu.$$

The roots of $x^2 + x + \mu = 0$ are $\dfrac{1 \pm \nu}{2} \in \mathbb{R} \cup \dfrac{1}{2} + \imath\mathbb{R}$.

Asymptotics for J

Assume $\mu \in (0,1/4)$ so that $\nu \in (0,1)$. Then, the solution is

$$J(t) = e^{-\frac{1+\nu}{2}t}\left(-\frac{1}{\nu}\int_0^t e^{-\frac{1-\nu}{2}\xi} G(\xi)\,\mathrm{d}\xi - \frac{1-\nu}{2\nu}J(0) - \frac{1}{\nu}J'(0)\right)$$
$$+ e^{-\frac{1-\nu}{2}t}\left(\frac{1}{\nu}\int_0^t e^{-\frac{1+\nu}{2}\xi} G(\xi)\,\mathrm{d}\xi + \frac{1+\nu}{2\nu}J(0) + \frac{1}{\nu}J'(0)\right)$$

Note that

$$e^{-\frac{1\pm\nu}{2}t}\left|\int_t^{\infty} e^{-\frac{1\mp\nu}{2}\xi} G(\xi)\,\mathrm{d}\xi\right| \leq \frac{4}{1\mp\nu}\|f\|_{\mathscr{C}^1}\, e^{-t}.$$

Asymptotics for J

Define

$$\mathcal{D}_\mu^\pm f(p) = \mp \frac{1}{\nu} \int_0^\infty e^{-\frac{1\mp\nu}{2}\xi} G_f(p,\xi)\,\mathrm{d}\xi \mp \frac{1\mp\nu}{2\nu} J_f(p,0) \mp \frac{1}{\nu} J_f'(p,0),$$

$$\left|\mathcal{D}_\mu^\pm f(p)\right| \leq \frac{6}{\nu(1-\nu)} \|f\|_{\mathscr{C}^1}.$$

Then,

$$J(t) = J_f(p,t) = e^{-\frac{1+\nu}{2}t} \mathcal{D}_\mu^+ f(p) + e^{-\frac{1-\nu}{2}t} \mathcal{D}_\mu^- f(p) + O\big(e^{-t}\big).$$

Ergodic averages

Let p_t denote $\varphi^{\mathbf{x}}_{\log t}(p)$.

Theorem

Let $f \in \mathscr{C}^2(M)$ with $\Box f = \mu f$ for some $\mu \in (0, 1/4)$. Then, there exist bounded functions $\mathcal{D}^{\pm}_{\mu} f$, with

$$\|\mathcal{D}^{\pm}_{\mu} f\|_{\infty} \leq \frac{6}{\nu(1-\nu)} \|f\|_{\mathscr{C}^1},$$

such that

$$\frac{1}{t}\int_0^t f \circ h_r(p)\,\mathrm{d}r = t^{-\frac{1+\nu}{2}} \mathcal{D}^{+}_{\mu} f(p_t) + t^{-\frac{1-\nu}{2}} \mathcal{D}^{-}_{\mu} f(p_t) + O\left(t^{-1}\right).$$

Proof. Follows from the formula for $J_f(p,t)$ and the relation $A_f(p,t) = J_f\left(p_t, \log t\right)$.

Ergodic averages

Theorem (Flaminio-Forni 2003, Bufetov-Forni 2014, R. 2023)

There exists an explicit constant $C_M \geq 0$ such that the following holds. Let $f \in \mathscr{C}^4(M)$. For every $\mu \in \mathrm{Spec}(\Box) \cap \mathbb{R}_{>0}$, there exist bounded functions $\mathcal{D}_\mu^{\pm} f$, with

$$\sum_{\mu \in \mathrm{Spec}(\Box) \cap \mathbb{R}_{>0}} \|\mathcal{D}_\mu^{\pm} f\|_\infty \leq C_M \|f\|_{\mathscr{C}^4},$$

such that for all $p \in M$ and $t \geq 1$ we have

$$\begin{aligned} A_f(p,t) = \int_M f \, \mathrm{d}\mathrm{vol} + \sum_{\mu \in \mathrm{Spec}(\Box)\mathbb{R}_{>0} \setminus \{\frac{1}{4}\}} t^{-\frac{1 \pm \nu}{2}} \mathcal{D}_\mu^{\pm} f(p_t) \\ + \mathbb{1}_{\mathrm{Spec}(\Box)}(1/4) \cdot \left(t^{-\frac{1}{2}} \mathcal{D}_{\frac{1}{4}}^{+} f(p_t) + t^{-\frac{1}{2}} \log t \, \mathcal{D}_{\frac{1}{4}}^{-} f(p_t) \right) \\ + O\left(\frac{1 + \log t}{t} \right). \end{aligned}$$

Ergodic averages

The proof uses the following ingredients:

- a spectral decomposition of $f \in L^2(M)$ into

$$f = \sum_{\mu \in \mathrm{Spec}(\square)} f_\mu, \qquad \text{where } f_\mu \in L^2(M) \text{ and } \square f_\mu = \mu\, f_\mu,$$

- Sobolev Embedding Theorem to ensure that $f_\mu \in \mathscr{C}^2(M)$,
- the fact that $\mathrm{Spec}(\square)$ is discrete and "explicit".

Corollary (Burger 1990)

There exist explicit constants $C_M \geq 0$ and $\nu_0 \in [0,1)$ such that for every $f \in \mathscr{C}^4(M)$ and every $p \in M$ we have

$$\left| A_f(p,t) - \int_M f \,\mathrm{d}\mathrm{vol} \right| \leq C_M \|f\|_{\mathscr{C}^4} t^{-\frac{1-\nu_0}{2}}.$$

Hölder regularity

Proposition

The functions $\mathcal{D}^{\pm}_{\mu} f$ are Hölder continuous with exponent $(1 \mp \Re \nu)/2$, except $\mathcal{D}^{+}_{\frac{1}{4}} f$ which has exponent $1/2 - \varepsilon$ for all $\varepsilon > 0$.

Proof (sketch). Fix $\mu \neq 1/4$. Recall that

$$\mathcal{D}^{\pm}_{\mu} f(p) = \mp \frac{1}{\nu} \int_0^{\infty} e^{-\frac{1 \mp \nu}{2} \xi} G_f(p, \xi) \, \mathrm{d}\xi \mp \frac{1 \mp \nu}{2\nu} J_f(p, 0) \mp \frac{1}{\nu} J'_f(p, 0).$$

The terms

$$J_f(p,0) = \int_0^1 f \circ h_r(p) \, \mathrm{d}r, \qquad \text{and} \qquad J'_f(p,0) = -\int_0^1 Xf \circ h_r(p) \, \mathrm{d}r$$

are $\mathscr{C}^1$-functions in p, hence we focus on the first term.

Hölder regularity

Let $p, q \in M$, and assume that

$$q = \varphi_s^{\mathbf{w}}(p) = p \exp(s\mathbf{w}), \qquad \text{for some} \qquad \mathbf{w} \in \mathfrak{g},$$

with $\|\mathbf{w}\|_\infty \leq 1$. It suffices to bound

$$\int_0^\infty e^{-a\xi} |G_f(p,\xi) - G_f(q,\xi)| \, \mathrm{d}\xi, \qquad \text{where} \qquad a = \frac{1 \mp \Re \nu}{2},$$
$$\text{and} \qquad G_f(p,\xi) = Vf \circ \varphi_{-\xi}^{\mathbf{x}}(p) - Vf \circ \varphi_{-\xi}^{\mathbf{x}} \circ h_1(p).$$

By the Mean-Value Theorem,

$$\begin{aligned} |G_f(p,\xi) - G_f(q,\xi)| &\leq \|f\|_{\mathscr{C}^2} \left(|D\varphi_{-\xi}^{\mathbf{x}}(\mathbf{w})| + |D\varphi_{-\xi}^{\mathbf{x}} \circ Dh_1(\mathbf{w})| \right) s \\ &\leq 6 \|f\|_{\mathscr{C}^2} e^{\xi} \, s. \end{aligned}$$

Hölder regularity

We obtained that

$$|G_f(p,\xi) - G_f(q,\xi)| \leq 6\|f\|_{\mathscr{C}^2} \min\{1, e^{\xi} s\}.$$

From this, it is a nice calculus exercise to show that

$$\int_0^{\infty} e^{-a\xi} |G_f(p,\xi) - G_f(q,\xi)| \, \mathrm{d}\xi \leq 6\|f\|_{\mathscr{C}^2} \max\left\{\frac{1}{1-a}, \frac{1}{a}\right\} s^a.$$

The case $\mu = 1/4$ is a slight modification of this proof.

A temporal distributional limit theorem

The case $\mu = 0$

Let $f \in \mathscr{C}^2(M)$ be such that $\Box f = 0$. In this case, $J(t) = J_f(p,t)$ satisfies

$$J''(t) + J'(t) = e^{-t} G(t).$$

We can easily find the solution

$$J(t) = \mathrm{const} - e^{-t} \int_0^t G(\xi)\,\mathrm{d}\xi + O(e^{-t}).$$

Recall that

$$J_f(p,t) = A_f(\varphi^{\mathbf{X}}_{-t}(p), e^t).$$

If we assume $\int_M f \,\mathrm{d}\mathrm{vol} = 0$, by unique ergodicity,

$$\|J_f(\cdot,t)\|_\infty = \|A_f(\cdot, e^t)\|_\infty \to 0, \qquad \text{so that const} = 0.$$

The case $\mu = 0$

Unpacking the definitions of J and G, we can prove the following formula.

Theorem

Let $f \in \mathscr{C}^2(M)$ be such that $\int_M f \, \mathrm{d}\mathrm{vol} = 0$ and $\Box f = 0$. Then, for every $p \in M$ and $t \geq 1$, we have

$$\int_0^t f \circ h_r(p) \, \mathrm{d}r = \int_0^{\log t} \left[Vf \circ \varphi_\xi^{\mathbf{X}} \circ h_t(p) - Vf \circ \varphi_\xi^{\mathbf{X}}(p) \right] \mathrm{d}\xi + O(1).$$

In other words, we related the integrals of f along a horocycle orbit of length t to the difference of integrals along two geodesic orbits of length $\log t$.

The case $\mu = 0$

Let us note that

$$\int_{\log t}^{\infty} \left[Vf \circ \varphi_\xi^{\mathbf{x}} \circ h_t(p) - Vf \circ \varphi_\xi^{\mathbf{x}}(p) \right] \mathrm{d}\xi = O(1).$$

Thus, for any $t \leq T$, it is also true that

$$\int_0^t f \circ h_r(p) \, \mathrm{d}r = \int_0^{\log T} \left[Vf \circ \varphi_\xi^{\mathbf{x}} \circ h_t(p) - Vf \circ \varphi_\xi^{\mathbf{x}}(p) \right] \mathrm{d}\xi + O(1).$$

If we call

$$C_T(p) := \int_0^{\log T} Vf \circ \varphi_\xi^{\mathbf{x}}(p) \, \mathrm{d}\xi,$$

we can rewrite

$$\frac{\int_0^t f \circ h_r(p) \, \mathrm{d}r - C_T(p)}{\sqrt{\log T}} = \frac{1}{\sqrt{\log T}} \int_0^{\log T} Vf \circ \varphi_\xi^{\mathbf{x}} \circ h_t(p) \, \mathrm{d}\xi + o(1).$$

A temporal CLT

We have

$$\frac{\int_0^t f \circ h_r(p)\,\mathrm{d}r - C_T(p)}{\sqrt{\log T}} = \frac{1}{\sqrt{\log T}} \int_0^{\log T} Vf \circ \varphi_\xi^{\mathbf{X}} \circ h_t(p)\,\mathrm{d}\xi + o(1).$$

- Imagine now taking $t \in [1, T]$ randomly uniformly.
- In the right-hand side we see an integral along a geodesic orbit of length $\log T$ where the point is chosen randomly uniformly on a horocycle orbit of length T.
- By unique ergodicity, horocycle orbits become equidistributed: uniform measures on long orbits converge weakly to the volume measure.
- Can we apply the CLT for the geodesic flow?

A temporal CLT

The previous "non-proof" can be turned into a real proof of the following result.

Theorem (Dolgopyat-Sarig 2017, Corso 2023)

Let $f \in \mathscr{C}^2(M)$ be such that $\int_M f \,\mathrm{d}\mathrm{vol} = 0$ and $\Box f = 0$. Assume that f is not a measurable coboundary for the horocycle flow. Then, there exists $\sigma > 0$ such that for every $p \in M$,

$$\frac{\int_0^t f \circ h_r(p)\,\mathrm{d}r - C_T(p)}{\sqrt{\log T}} \to \mathcal{N}(0, \sigma), \qquad t \sim \mathcal{U}[1, T]$$

in distribution, as $t \to \infty$.