Ergodic and mixing properties of horocycle flows and their time-changes

Simons Semester – Lecture 3

Davide Ravotti

University of Vienna

29 May 2023

Plan for today

Today we will study the mixing properties of the horocycle flows. We will look at:

- the *mixing via shearing* method (using geodesic curves),
- the equidistribution of arbitrary homogeneous curves,
- Ratner's result on the rate of decay of correlations.

Mixing via shearing

The setting

- $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma \leq G$ is discrete,
- $M = \Gamma \backslash G$ is compact,
- vol is the Haar measure on M, normalized so that $\mathrm{vol}(M) = 1$.
- $\mathbf{u} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}$ generates the horocycle flow

$$h_t = \varphi_t^{\mathbf{u}} \colon \Gamma g \mapsto \Gamma g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

- $\mathbf{x} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \in \mathfrak{g}$ generates the geodesic flow

$$\varphi_t^{\mathbf{x}} \colon \Gamma g \mapsto \Gamma g \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}.$$

Mixing via shearing

Saying that (h_t) is mixing means that for any measurable set A with $\mu(A) > 0$ and any $f \in L^\infty(M)$ we have

$$\frac{1}{\mu(A)} \int_A f \circ h_t \,\mathrm{d}\,\mathrm{vol} \to \int_M f \,\mathrm{d}\,\mathrm{vol}, \qquad \text{as } t \to \infty.$$

Instead, we now look at (smooth) arcs $\gamma \colon [0, \sigma] \to M$.

The smooth arc $\gamma \colon [0, \sigma] \to M$ equidistributes under (h_t) if for every $f \in L^\infty(M)$ we have

$$\frac{1}{\sigma} \int_0^\sigma f \circ h_t \circ \gamma(s) \,\mathrm{d}s \to \int_M f \,\mathrm{d}\,\mathrm{vol}, \qquad \text{as } t \to \infty.$$

Mixing via shearing

Let $\mathbf{w} \in \mathfrak{g} \setminus \{0\}$, then we let

$$\gamma^{\mathbf{w}}_{p,\sigma}(s) = \varphi^{\mathbf{w}}_s(p) = p \exp(s\mathbf{w}), \qquad \text{for } s \in [0, \sigma].$$

Lemma

Let $f \in L^\infty(M)$ with $\int_M f \,\mathrm{d}\mathrm{vol} = 0$, and let $g \in \mathscr{C}^1(M)$. Then, for any $S \in (0,1]$, we have

$$\left| \int_M f \circ h_t \cdot g \,\mathrm{d}\mathrm{vol} \right| \leq 2 \|g\|_{\mathscr{C}^1} \cdot \frac{1}{S} \sup_{\sigma \in (0,S]} \left\| \int_0^\sigma f \circ h_t \circ \varphi^{\mathbf{w}}_s \,\mathrm{d}s \right\|_2 .$$

Equidistribution of arcs $\gamma^{\mathbf{w}}_{p,\sigma}$ under (h_t) for "most" points p implies mixing.

Proof of the Lemma

By measure invariance,

$$\int_M f \circ h_t \cdot g \,\mathrm{d}\mathrm{vol} = \int_M (f \circ h_t \cdot g) \circ \varphi_s^{\mathbf{w}} \,\mathrm{d}\mathrm{vol},$$

for all $s \in \mathbb{R}$. Therefore, given $S > 0$,

$$\int_M f \circ h_t \cdot g \,\mathrm{d}\mathrm{vol} = \frac{1}{S} \int_0^S \int_M (f \circ h_t \cdot g) \circ \varphi_\sigma^{\mathbf{w}} \,\mathrm{d}\mathrm{vol}\,\mathrm{d}\sigma.$$

Integrating by parts,

$$\int_0^S (f \circ h_t \circ \varphi_\sigma^{\mathbf{w}}) \cdot (g \circ \varphi_\sigma^{\mathbf{w}}) \,\mathrm{d}\sigma = \left(\int_0^S f \circ h_t \circ \varphi_\sigma^{\mathbf{w}} \,\mathrm{d}\sigma \right) \cdot (g \circ \varphi_S^{\mathbf{w}}) - \int_0^S \left(\int_0^\sigma f \circ h_t \circ \varphi_s^{\mathbf{w}} \,\mathrm{d}s \right) \cdot Wg \circ \varphi_\sigma^{\mathbf{w}} \,\mathrm{d}\sigma.$$

Proof of the Lemma

We obtained

$$\int_M f\circ h_t \cdot g \,\mathrm{d}\mathrm{vol} = \frac{1}{S}\int_M \left(\int_0^S f\circ h_t\circ \varphi^{\mathbf{w}}_\sigma \,\mathrm{d}\sigma\right)\cdot (g\circ \varphi^{\mathbf{w}}_S)\,\mathrm{d}\mathrm{vol} \\ - \frac{1}{S}\int_M\int_0^S \left(\int_0^\sigma f\circ h_t\circ \varphi^{\mathbf{w}}_s \,\mathrm{d}s\right)\cdot Wg\circ \varphi^{\mathbf{w}}_\sigma \,\mathrm{d}\sigma\,\mathrm{d}\mathrm{vol}\,.$$

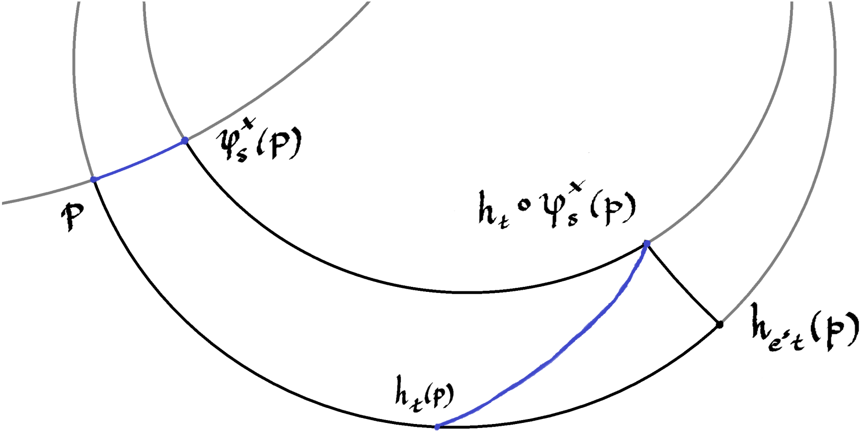
By Cauchy-Schwarz,

$$\left|\int_M f\circ h_t \cdot g\,\mathrm{d}\mathrm{vol}\right| \leq \|g\|_{\mathscr{C}^1}\cdot \frac{1}{S}\left\|\int_0^S f\circ h_t\circ\varphi^{\mathbf{w}}_\sigma\,\mathrm{d}\sigma\right\|_2 \\ + \|g\|_{\mathscr{C}^1}\cdot \sup_{\sigma\in(0,S]}\left\|\int_0^\sigma f\circ h_t\circ\varphi^{\mathbf{w}}_s\,\mathrm{d}s\right\|_2 \\ \leq 2\|g\|_{\mathscr{C}^1}\cdot\frac{1}{S}\sup_{\sigma\in(0,S]}\left\|\int_0^\sigma f\circ h_t\circ\varphi^{\mathbf{w}}_s\,\mathrm{d}s\right\|_2 .$$

Shearing of geodesic segments

We look at the case $\mathbf{w} = \mathbf{x}$. Recall that

$$\varphi_s^{\mathbf{x}} \circ h_{e^s t}(p) = h_t \circ \varphi_s^{\mathbf{x}}(p) \qquad \text{for any } p \in M.$$



Shearing of geodesic segments

Let $\gamma^{\mathbf{x}}_{p,\sigma}(s) = \varphi^{\mathbf{x}}_s(p)$, for $s \in [0,\sigma]$, be a geodesic segment, with $\sigma \leq 1$. Then,

$$h_t \circ \gamma^{\mathbf{x}}_{p,\sigma}(s) = h_t \circ \varphi^{\mathbf{x}}_s(p) = \varphi^{\mathbf{x}}_s \circ h_{e^s t}(p) \approx h_{e^s t}(p), \qquad \text{for } s \in [0,\sigma].$$

More precisely, if $f \in \mathscr{C}^1(M)$, then we have

$$\begin{aligned} &\left| \int_0^\sigma f \circ h_t \circ \varphi^{\mathbf{x}}_s(p) \,\mathrm{d}s - \int_0^\sigma f \circ h_{e^s t}(p) \,\mathrm{d}s \right| \\ &\qquad \leq \int_0^\sigma |f \circ \varphi^{\mathbf{x}}_s - f| \circ h_{e^s t}(p) \,\mathrm{d}s \leq \|Xf\|_\infty \cdot \sigma^2. \end{aligned}$$

Moreover,

$$\left| \int_0^\sigma f \circ h_{e^s t}(p) \,\mathrm{d}s - \int_0^\sigma f \circ h_{(s+1)t}(p) \,\mathrm{d}s \right| \leq \|Uf\|_\infty \cdot \sigma^3 t.$$

Shearing of geodesic segments

A change of variables gives us

$$\int_0^{\sigma} f \circ h_{(s+1)t}(p)\,\mathrm{d}s = \frac{1}{t}\int_0^{\sigma t} f \circ h_s(h_t(p))\,\mathrm{d}s.$$

thus,

$$\left| \int_0^{\sigma} f \circ h_t \circ \varphi_s^{\mathsf{X}}(p)\,\mathrm{d}s - \frac{1}{t}\int_0^{\sigma t} f \circ h_s(h_t(p))\,\mathrm{d}s \right| \leq \|f\|_{\mathscr{C}^1} \cdot (\sigma^2 + \sigma^3 t).$$

The remaining term to bound is an ergodic integral at time σt.

Shearing of geodesic segments

In Lecture 2 we saw that, if $f \in \mathscr{C}^4(M)$ has zero average, then

$$\left\| \int_0^{\sigma t} f \circ h_s \, \mathrm{d}s \right\|_\infty \leq C_M \|f\|_{\mathscr{C}^4} (\sigma t)^{\frac{1+\nu_0}{2}},$$

for some explicit $\nu_0 \in [0,1)$, whenever $\sigma t \geq 1$.
Thus, we proved that for every $t \geq 1$ and $0 < \sigma \leq 1$ we have

$$\left| \int_0^{\sigma} f \circ h_t \circ \varphi_s^{\mathbf{x}}(p) \, \mathrm{d}s \right| \leq C_M \|f\|_{\mathscr{C}^4} \cdot \left(t^{-1} \min \left\{ (\sigma t)^{\frac{1+\nu_0}{2}}, \sigma t \right\} + \sigma^2 + \sigma^3 t \right).$$

This implies that

$$\sup_{\sigma \in (0, t^{-1/2}]} \left| \int_0^{\sigma} f \circ h_t \circ \varphi_s^{\mathbf{x}}(p) \, \mathrm{d}s \right| \leq C_M \|f\|_{\mathscr{C}^4} t^{-\frac{3-\nu_0}{4}}.$$

Mixing via shearing

From the mixing via shearing lemma, choosing $S = t^{-1/2}$, for any $f \in \mathscr{C}^4(M)$ with $\int_M f \,\mathrm{d}\mathrm{vol} = 0$ and $g \in \mathscr{C}^1(M)$, we have

$$\left| \int_M f \circ h_t \cdot g \,\mathrm{d}\mathrm{vol} \right| \leq 2 \|g\|_{\mathscr{C}^1} \cdot t^{1/2} \sup_{\sigma \in (0, t^{-1/2}]} \left\| \int_0^{\sigma} f \circ h_t \circ \varphi_s^{\mathbf{x}} \,\mathrm{d}s \right\|_2$$
$$\leq 2 C_M \|g\|_{\mathscr{C}^1} \|f\|_{\mathscr{C}^4} t^{-\frac{1-\nu_0}{4}}.$$

Theorem

The horocycle flow on M is mixing with polynomial rates: there exist $C_M > 0$ and $\eta \in (0,1]$ such that for every $f, g \in \mathscr{C}^4(M)$ we have

$$\left| \int_M f \circ h_t \cdot g \,\mathrm{d}\mathrm{vol} - \left(\int_M f \,\mathrm{d}\mathrm{vol} \right) \left(\int_M g \,\mathrm{d}\mathrm{vol} \right) \right| \leq C_M \|g\|_{\mathscr{C}^1} \|f\|_{\mathscr{C}^4} t^{-\eta}.$$

What is the optimal exponent?

- By the mixing via shearing lemma, the speed of mixing was related to the equidistribution of curves $\gamma^{\mathbf{w}}_{p,\sigma}(s) = \varphi^{\mathbf{w}}_s(p)$.
- We chose $\mathbf{w} = \mathbf{x}$ (i.e., geodesic arcs) and we proved that $h_t \circ \gamma^{\mathbf{x}}_{p,\sigma}$ approximate a horocycle orbit of length $t\sigma$.
- By unique ergodicity, $h_t \circ \gamma^{\mathbf{x}}_{p,\sigma}$ becomes equidistributed as $t \to \infty$.
- Note that this approach will give a speed of mixing which at best matches the speed of equidistribution of horocycle orbits.
- Do other curves equidistribute faster?

Equidistribution of homogeneous arcs

Equidistribution of homogeneous curves

Recall that the equidistribution of a horocycle average at time t is $O(t^{-\frac{1-\nu_0}{2}})$, for some $\nu_0 \in [0,1)$.

Theorem (Bufetov-Forni 2014)

Let $\mathbf{w} = a\mathbf{u} + b\mathbf{x}$, with $b \neq 0$. For every $S > 0$ there exists a constant $C = C(\mathbf{w}, S, M)$ such that for all $f \in \mathscr{C}^6(M)$ with $\int_M f \,\mathrm{d}\mathrm{vol} = 0$, for all $p \in M$ and for all $t \geq 1$ we have

$$\sup_{\sigma \in (0,S]} \left| \int_0^\sigma f \circ h_t \circ \varphi_s^{\mathbf{w}}(p) \,\mathrm{d}s \right| \leq C \|f\|_{\mathscr{C}^6} t^{-\frac{1-\nu_0}{2}} (1 + \log t).$$

Small note: in the case $\nu_0 > 0$, the factor $(1 + \log t)$ is not present.

Equidistribution of homogeneous curves

Theorem (R. 2020)

Let $\mathbf{w} = a\mathbf{u} + b\mathbf{x} + c\mathbf{v}$, with $c \neq 0$. For every $S > 0$ there exists a constant $C = C(\mathbf{w}, S, M)$ such that for all $f \in \mathscr{C}^6(M)$ with $\int_M f \,\mathrm{d}\mathrm{vol} = 0$, for all $p \in M$ and for all $t \geq 1$ we have

$$\sup_{\sigma \in (0,S]} \left| \int_0^\sigma f \circ h_t \circ \varphi_s^{\mathbf{w}}(p) \,\mathrm{d}s \right| \leq C \|f\|_{\mathscr{C}^6} t^{-(1-\nu_0)} (1 + \log t).$$

Equidistribution of homogeneous curves

- Homogeneous arcs transverse to the foliation tangent to $\langle X, U \rangle$ —namely, the weak-stable leaves of the geodesic flow— equidistribute faster than geodesic arcs.
- By the mixing via shearing lemma, the rate of mixing of the horocycle flow is at least $O(t^{-(1-\nu_0)})$; the exponent is double the exponent coming from equidistribution of orbits.

Key observation

Let us look at the case

$$\gamma^{\mathbf{v}}_{p,\sigma}(s) = \varphi^{\mathbf{v}}_s(p) = p \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \qquad \text{for } s \in [0, \sigma].$$

The key observation is that

$$\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/t & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/t & 1 \end{pmatrix} \begin{pmatrix} 1 & -st^2 \\ 0 & 1 \end{pmatrix}.$$

which means

$$\varphi^{\mathbf{v}}_{-1/t} \circ h_t \circ \gamma^{\mathbf{v}}_{p,\sigma}(s) = h_{-st^2}(p_t), \qquad \text{where } p_t = \varphi^{\mathbf{v}}_{-1/t} \circ h_t(p).$$

Key observation

From

$$\varphi^{\mathbf{v}}_{-1/t} \circ h_t \circ \gamma^{\mathbf{v}}_{p,\sigma}(s) = h_{-st^2}(p_t), \qquad \text{where } p_t = \varphi^{\mathbf{v}}_{-1/t} \circ h_t(p),$$

roughly speaking, we deduce

$$\frac{1}{\sigma}\int_0^{\sigma} h_t \circ \gamma^{\mathbf{v}}_{p,\sigma}(s)\,\mathrm{d}s \approx \frac{1}{\sigma}\int_0^{\sigma} h_{-st^2}(p_t)\,\mathrm{d}s = \frac{1}{\sigma t^2}\int_0^{\sigma t^2} h_{-s}(p_t)\,\mathrm{d}s.$$

In other words, the segment $h_t \circ \gamma^{\mathbf{v}}_{p,\sigma}(s)$ is "very close" to a horocycle orbit of length σt^2, parametrized with constant speed.

Ratner's Theorem on mixing rates

Our observables

Recall from Lecture 1 that the Casimir operator $\Box$ was defined as

$$\Box = R^2 - X^2 - Y^2 = -X^2 + X - U^2 - 2UR,$$

where $\mathbf{r} = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ is the generator of SO(2).
We consider a functions $f \in \mathscr{C}^2(M)$ such that

$$\Box f = \mu f, \qquad Rf = 0,$$

for some $\mu > 0$.

Our observables

- $Rf = 0$ means that f is $\mathrm{SO}(2)$-invariant and hence is defined on $M/\mathrm{SO}(2)$ or, equivalently, defined on $G/\mathrm{SO}(2)$ and invariant by Γ.
- In Lecture 1, we saw that $G/\mathrm{SO}(2)$ can be identified with the hyperbolic plane $\mathscr{H}$, so f is a function on $\mathscr{H}$ invariant by the action of Γ. That is, f is a function on the hyperbolic surface $S = \Gamma \backslash \mathscr{H}$.
- For positive $\mu > 0$, saying that f satisfies $\Box f = \mu f$ and $Rf = 0$ is equivalent to asking that $\Delta_S f = \mu f$, i.e., f is an eigenfunction of the Laplacian on S.

The function K

We want to compute the self-correlations

$$K(t) := \langle f \circ h_t, f \rangle = \int_M f \circ h_t \cdot \overline{f} \, \mathrm{d}\mathrm{vol}.$$

Note that, for any $j \geq 0$,

$$K^{(j)}(t) = \frac{\mathrm{d}^j}{\mathrm{d}t^j} \langle f \circ h_t, f \rangle = \langle \frac{\mathrm{d}^j}{\mathrm{d}t^j} f \circ h_t, f \rangle = \langle U^j f \circ h_t, f \rangle.$$

We also define

$$Q(t) := \langle Xf \circ h_t, f \rangle.$$

The function K

We have

$$0 = -\langle f \circ h_t, Rf \rangle = \langle R(f \circ h_t), f \rangle = \langle [Dh_t(R) f] \circ h_t, f \rangle.$$

From Lecture 1, we can compute

$$Dh_t(R) \simeq \mathrm{Ad}(\exp(t\mathbf{u}))\mathbf{r} = \mathbf{r} - t\mathbf{x} - \frac{t^2}{2}\mathbf{u}.$$

Therefore

$$\begin{aligned} 0 &= \langle Rf \circ h_t, f \rangle - t \langle Xf \circ h_t, f \rangle - \frac{t^2}{2} \langle Uf \circ h_t, f \rangle \\ &= -tQ(t) - \frac{t^2}{2} K'(t). \end{aligned}$$

The function K

For all $t \geq 1$, we obtained

$$Q(t) = -\frac{t}{2} K'(t).$$

This implies that

$$Q'(t) = -\frac{1}{2} K'(t) - \frac{t}{2} K''(t).$$

Remark: since $K'(t) = \langle Uf \circ h_t, f \rangle$ and $Q'(t) = \langle UXf \circ h_t, f \rangle$, by the Cauchy-Schwarz Inequality,

$$|K'(t)| \leq \|f\|^2_{\mathscr{C}^1}, \qquad \text{and}$$
$$|K''(t)| \leq \frac{1}{t}(|K'(t)| + 2|Q'(t)|) \leq \frac{3}{t}\|f\|^2_{\mathscr{C}^2}.$$

The function $Q(t)$

We do the same for $Q(t)$ instead of $K(t)$ and we get

$$\begin{aligned} 0 &= -\langle Xf \circ h_t, Rf \rangle = \langle R(Xf \circ h_t), f \rangle = \langle [Dh_t(R)Xf] \circ h_t, f \rangle \\ &= \langle RXf \circ h_t, f \rangle - t\langle X^2 f \circ h_t, f \rangle - \frac{t^2}{2}\langle UXf \circ h_t, f \rangle. \end{aligned}$$

Since, from Lecture 1, $\mathbf{r}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{r} = [\mathbf{r},\mathbf{x}] = \frac{1}{2}\mathbf{y} = \frac{1}{2}(\mathbf{r}+\mathbf{u})$, we can write

$$\begin{aligned} 0 =& \langle XRf \circ h_t, f \rangle + \frac{1}{2}\langle Rf \circ h_t, f \rangle + \frac{1}{2}\langle Uf \circ h_t, f \rangle \\ & - t\langle X^2 f \circ h_t, f \rangle - \frac{t^2}{2}\langle UXf \circ h_t, f \rangle \\ =& \frac{1}{2}K'(t) - t\langle X^2 f \circ h_t, f \rangle - \frac{t^2}{2}Q'(t). \end{aligned}$$

The equation to solve

We proved that

$$Q(t) = -\frac{t}{2}K'(t), \qquad Q'(t) = -\frac{1}{2}K'(t) - \frac{t}{2}K''(t),$$
$$\langle X^2 f \circ h_t, f\rangle = \frac{1}{2t}K'(t) - \frac{t}{2}Q'(t).$$

Since $\Box f = \mu f$ we also have

$$\begin{aligned}\mu K(t) &= \langle \mu f \circ h_t, f\rangle = \langle \Box f \circ h_t, f\rangle \\ &= \langle (-X^2 + X - U^2 - 2UR) f \circ h_t, f\rangle \\ &= -\langle X^2 f \circ h_t, f\rangle + Q(t) - K''(t).\end{aligned}$$

The equation to solve

Plugging everything together, we find

$$t^2 K''(t) + 3tK'(t) + 4\mu K(t) = -4K''(t) - \frac{2}{t}K'(t).$$

Recalling the bounds we had for $|K'(t)|$ and for $|K''(t)|$, we proved the following result.

Theorem

Let $f \in \mathscr{C}^2(M)$ be such that $Rf = 0$ and $\Box f = \mu f$ for some $\mu > 0$. Then the self-correlations $K(t) = \langle f \circ h_t, f \rangle$ satisfy

$$t^2 K''(t) + 3tK'(t) + 4\mu K(t) = P(t), \qquad \text{for all } t \geq 1,$$

where $P(t)$ is such that $|P(t)| \leq 14\|f\|_{\mathscr{C}^2} t^{-1}$.

The solution

Assume $\mu \neq 1/4$. Then, the previous equation can be solved exactly: if we let $\nu \in \mathbb{R}_{>0} \cup i\mathbb{R}_{>0}$ be such that $\nu^2 = 1 - 4\mu$, we get

$$K(t) = \frac{t^{-1+\nu}}{2\nu} \int_1^t r^{-\nu} P(r)\,\mathrm{d}r - \frac{t^{-1-\nu}}{2\nu} \int_1^t r^{\nu} P(r)\,\mathrm{d}r + c_1 t^{-1+\nu} + c_2 t^{-1-\nu},$$

where the constants c_1, c_2 are determined by the initial conditions $K(1)$ and $K'(1)$.
Recalling $P(t) = O(t^{-1})$, we can rewrite

$$K(t) = \frac{t^{-1+\nu}}{2\nu} \left(\int_1^\infty r^{-\nu} P(r)\,\mathrm{d}r + c_1 \right) + O(t^{-1}).$$

The solution

Theorem

Let $f \in \mathscr{C}^2(M)$ be such that $Rf = 0$ and $\Box f = \mu f$ for some $\mu > 0$, $\mu \neq 1/4$. Then, there exists a constant $A = A(f)$ such that

$$|\langle f \circ h_t, f\rangle - At^{-1+\nu}| = O(t^{-1}).$$

In particular,

$$|\langle f \circ h_t, f\rangle| = O(t^{-1+\Re\nu}).$$

Exercise: generalise the theorem above to $\langle f \circ h_t, g\rangle$, where $f, g \in \mathscr{C}^2(M)$ are such that

$$\Box f = \mu f, \qquad Rf = i\,n f,$$
$$\Box g = \mu g, \qquad Rg = i\,m g,$$

for some $\mu \in \mathbb{R}$ and $n, m \in \mathbb{Z}$.

Ratner's Theorem

Theorem (Ratner 1987)

There exists $\nu_0 \in [0,1)$ such that the following holds. Let f, g be such that $R^3 f$ and $R^3 g$ exist and are continuous. Then, there exists a constant $A = A(f,g)$ such that

$$\left| \int_M f \circ h_t \cdot g \, \mathrm{d}\mathrm{vol} - \left(\int_M f \, \mathrm{d}\mathrm{vol} \right) \left(\int_M g \, \mathrm{d}\mathrm{vol} \right) \right| \leq A t^{-1+\nu},$$

for all $t \geq 1$.

Ratner's Theorem

The proof uses a spectral decomposition of $f \in L^2(M)$ into

$$f = \sum_{\mu \in \mathrm{Spec}(\Box)} \sum_{n \in I_\mu} f_{\mu,n},$$

where $I_\mu \subseteq \mathbb{Z}$ and $f_{\mu,n} \in \mathscr{C}^\infty(M)$ satisfies $\Box f_{\mu,n} = \mu\, f_{\mu,n}$ and $R f_{\mu,n} = i n f_{\mu,n}$. All these functions $f_{\mu,n}$ are mutually orthogonal.