Ergodic and mixing properties of horocycle flows and their time-changes

Simons Semester – Lecture 4

Davide Ravotti

University of Vienna

29 May 2023

Plan for today

Today we will introduce and study time-changes of horocycle flows. In particular, we will

- define what they are and look at their basic properties,
- prove ergodicity and mixing.

Time-changes of horocycle flows

The setting

- $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma \leq G$ is discrete,
- $M = \Gamma \backslash G$ is compact,
- vol is the Haar measure on M, normalized so that $\mathrm{vol}(M) = 1$.
- $\mathbf{u} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}$ generates the horocycle flow

$$h_t = \varphi_t^{\mathbf{u}} \colon \Gamma g \mapsto \Gamma g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

- $\mathbf{x} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \in \mathfrak{g}$ generates the geodesic flow

$$\varphi_t^{\mathbf{x}} \colon \Gamma g \mapsto \Gamma g \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}.$$

Time-changes

A time-change of $(h_t)_{t\in\mathbb{R}}$ is a smooth perturbation of U which preserves the orbits.

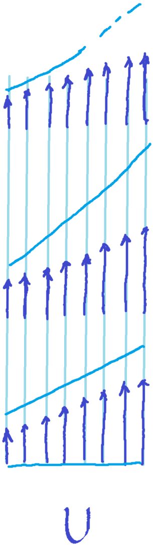
Let $\alpha\colon M \to \mathbb{R}_{>0}$. We will assume

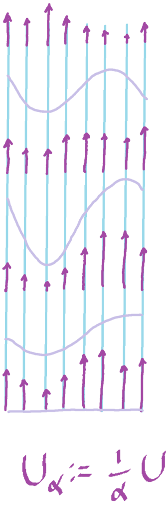
$$\int_M \alpha \, \mathrm{d}\mathrm{vol} = 1.$$

Definition

The *time-change* of $(h_t)_{t\in\mathbb{R}}$ induced by α is the smooth flow $(h_t^\alpha)_{t\in\mathbb{R}}$ generated by the vector field $U_\alpha := \alpha^{-1} U$.

Time-changes





Time-changes

Since orbits of $(h_t^\alpha)_{t\in\mathbb{R}}$ and of $(h_t)_{t\in\mathbb{R}}$ are the same, we can write

$$h_t^\alpha(p) = h_{\tau(t,p)}(p),$$

for some $\tau(t,p) \in \mathbb{R}$. By definition,

$$\begin{aligned} h_{\tau(t+r,p)}(p) = h_{t+r}^\alpha(p) = h_t^\alpha(h_r^\alpha(p)) &= h_{\tau(t,h_r^\alpha(p))} \circ h_{\tau(r,p)}(p) \\ &= h_{\tau(t,h_r^\alpha(p))+\tau(r,p)}(p). \end{aligned}$$

The function $\tau(t,p)$ is an additive cocycle over the flow $(h_t^\alpha)_{t\in\mathbb{R}}$, namely

$$\tau(t+r,p)(p) = \tau(t, h_r^\alpha(p)) + \tau(r,p),$$

for all $p \in M$ and $t, r \in \mathbb{R}$.

The cocycle $\tau(t,p)$

Lemma

The cocycle $\tau(t,p)$ is uniquely defined by the equality

$$t = \int_0^{\tau(t,p)} \alpha \circ h_r(p) \, \mathrm{d}r,$$

for all $p \in M$ and $t \in \mathbb{R}$.

Proof. Differentiating both sides of $h_r^\alpha(p) = h_{\tau(r,p)}(p)$ with respect to r, we get

$$\alpha^{-1}(h_r^\alpha(p)) \cdot U = \frac{\mathrm{d}}{\mathrm{d}r} h_r^\alpha(p) = \frac{\mathrm{d}}{\mathrm{d}r} h_{\tau(r,p)}(p) = \frac{\mathrm{d}}{\mathrm{d}r} \tau(r,p) \cdot U.$$

The cocycle $\tau(t, p)$

We can rewrite the previous equality as

$$1 = \frac{\mathrm{d}}{\mathrm{d}r}\tau(r,p) \cdot \alpha(h_{\tau(r,p)}(p)).$$

Integrating both sides from 0 to t, we conclude

$$t = \int_0^t \frac{\mathrm{d}}{\mathrm{d}r}\tau(r,p) \cdot \alpha \circ h_{\tau(r,p)}(p)\,\mathrm{d}r = \int_0^{\tau(t,p)} \alpha \circ h_r(p)\,\mathrm{d}r.$$

Uniqueness follows from the fact that $\alpha > 0$ and hence the equation above has at most one solution.

The cocycle $\tau(t, p)$

We want to estimate the difference in the time-parametrizations.

Proposition

There exists $\gamma \in (0,1]$ and there exists $A > 0$ depending on α such that for all $p \in M$ and $t \geq A$ we have

$$|\tau(t,p) - t| \leq A t^{1-\gamma}.$$

Proof.
Since M is compact, there exists $A' \geq 1$ such that

$$\frac{1}{A'} \leq \alpha(p) \leq A', \qquad \text{for all } p \in M.$$

Therefore,

$$\frac{\tau(t,p)}{A'} \leq \int_0^{\tau(t,p)} \alpha \circ h_r(p) \, \mathrm{d}r = t \leq A' \tau(t,p).$$

The cocycle $\tau(t, p)$

From what we proved about the ergodic integrals (see Lecture 2), we know that, up to increasing $A' > 0$,

$$\left| T - \int_0^T \alpha \circ h_r(p) \, \mathrm{d}r \right| \leq A' T^{1-\gamma}, \qquad \text{for all } T \geq 1 \text{ and } p \in M.$$

When $t \geq A'$, then $\tau(t, p) \geq 1$ so that, substituting $T = \tau(t, p)$, we conclude

$$\begin{aligned} |\tau(t,p) - t| = \left| \tau(t,p) - \int_0^{\tau(t,p)} \alpha \circ h_r(p) \, \mathrm{d}r \right| \leq A' \tau(t,p)^{1-\gamma} \\ \leq (A')^{2-\gamma} t^{1-\gamma}, \end{aligned}$$

which proves the result.

Ergodicity and mixing

Ergodicity

Let vol^α be the measure obtained by integrating the differential form

$$\mathrm{d}\,\mathrm{vol}^\alpha = \alpha\, \mathrm{d}\,\mathrm{vol}.$$

By our normalization, vol^α is a probability measure on M.

Lemma

The measure vol^α is invariant and ergodic for $(h_t^\alpha)_{t\in\mathbb{R}}$.

Ergodicity

Proof (sketch).

- The invariance claim is equivalent to

$$\mathscr{L}_{\alpha^{-1}U}(\alpha\, \mathrm{d}\mathrm{vol}) = 0,$$

- By Cartan's "magic formula",

$$\mathscr{L}_{\alpha^{-1}U}(\alpha\, \mathrm{d}\mathrm{vol}) = \mathscr{L}_{U}(\mathrm{d}\mathrm{vol}) = 0,$$

  where the last equality is equivalent to the invariance of vol under the horocycle flow.

- Ergodicity of the time-change follows from the fact that the flows $(h_t)_{t\in\mathbb{R}}$ and $(h_t^{\alpha})_{t\in\mathbb{R}}$ have the same invariant sets and the measures vol and vol^{α} have the same null sets.

Ergodicity

Let us now look at the ergodic averages. Recall that we proved that

$$\frac{\mathrm{d}}{\mathrm{d}r}\tau(r,p) = \alpha^{-1}(h_{\tau(r,p)}(p)).$$

Let $f \in \mathscr{C}^4(M)$, and let $t \geq A \geq 1$. We have

$$\begin{aligned}\frac{1}{t}\int_0^t f \circ h_r^{\alpha}(p)\,\mathrm{d}r &= \frac{1}{t}\int_0^t (\alpha f)\circ h_{\tau(r,p)}(p)\,\frac{\mathrm{d}}{\mathrm{d}r}\tau(r,p)\,\mathrm{d}r \\ &= \frac{1}{t}\int_0^{\tau(t,p)} (\alpha f)\circ h_r(p)\,\mathrm{d}r.\end{aligned}$$

From the results of Lecture 2, we have

$$\left| \frac{1}{\tau(t,p)}\int_0^{\tau(t,p)} (\alpha f)\circ h_r(p)\,\mathrm{d}r - \int_M \alpha f \,\mathrm{d}\mathrm{vol} \right| \leq A\|\alpha f\|_{\mathscr{C}^4}\,\tau(t,p)^{-\gamma}.$$

Ergodicity

We obtained

$$\left| \frac{1}{t} \int_0^t f \circ h_r^\alpha(p) \, \mathrm{d}r - \int_M f \, \mathrm{d}\mathrm{vol}^\alpha \right| \leq \frac{\tau(t,p)}{t} A \|\alpha f\|_{\mathscr{C}^4} \, \tau(t,p)^{-\gamma}.$$

Up to increasing the constant $A \geq 1$, we deduce the following theorem.

Theorem (Forni-Ulcigrai 2012)

Let $(h_t^\alpha)_{t \in \mathbb{R}}$ be a time-change of the horocycle flow on M. There exists $A > 0$ such that, for all $f \in \mathscr{C}^4(M)$ and for all $t \geq A$, we have

$$\left| \frac{1}{t} \int_0^t f \circ h_r^\alpha(p) \, \mathrm{d}r - \int_M f \, \mathrm{d}\mathrm{vol}^\alpha \right| \leq A \|\alpha f\|_{\mathscr{C}^4} \, t^{-\gamma}.$$

Mixing

In order to prove mixing, we resort to the "mixing via shearing" method we used in Lecture 3.
The same argument we have already used gives us the following lemma.

Lemma

Let $f, g \in \mathscr{C}^1(M)$ with $\int_M f \,\mathrm{d}\mathrm{vol} = 0$. Then, for any $S \in (0,1]$, we have

$$\left| \int_M f \circ h_t^\alpha \cdot g \,\mathrm{d}\mathrm{vol}^\alpha \right| \leq 3 \|\alpha\|_{\mathscr{C}^1} \|g\|_{\mathscr{C}^1} \cdot \frac{1}{S} \sup_{\sigma \in (0,S]} \left\| \int_0^\sigma f \circ h_t^\alpha \circ \varphi_s^{\mathsf{x}} \,\mathrm{d}s \right\|_2 .$$

A sketch of the proof

Proof (sketch).
By invariance of vol for the geodesic flow, given $S > 0$, we have

$$\begin{aligned}\int_M f \circ h_t^\alpha \cdot g \, \mathrm{d}\mathrm{vol}^\alpha &= \int_M f \circ h_t^\alpha \cdot \alpha g \, \mathrm{d}\mathrm{vol} \\ &= \frac{1}{S} \int_0^S \int_M (f \circ h_t^\alpha \circ \varphi_\sigma^{\mathbf{x}}) \cdot (\alpha g) \circ \varphi_\sigma^{\mathbf{x}} \, \mathrm{d}\mathrm{vol} \, \mathrm{d}\sigma.\end{aligned}$$

Integrating by parts and using the Cauchy-Schwarz Inequality, we obtain

$$\left| \int_M f \circ h_t^\alpha \cdot g \, \mathrm{d}\mathrm{vol}^\alpha \right| \leq (\|\alpha g\|_\infty + \|X(\alpha g)\|_\infty) \cdot \frac{1}{S} \sup_{\sigma \in (0,S]} \left\| \int_0^\sigma f \circ h_t^\alpha \circ \varphi_s^{\mathbf{x}} \, \mathrm{d}s \right\|_2,$$

which completes the proof.

Mixing via shearing

We reduced the problem to estimate

$$\frac{1}{S}\sup_{\sigma\in(0,S]}\left\|\int_0^\sigma f\circ h_t^\alpha\circ\varphi_s^{\mathbf{x}}\,\mathrm{d}s\right\|_2.$$

Since

$$h_t^\alpha\circ\varphi_s^{\mathbf{x}}(p)=h_{\tau(t,\varphi_s^{\mathbf{x}}(p))}\circ\varphi_s^{\mathbf{x}}(p)=\varphi_s^{\mathbf{x}}\circ h_{e^s\,\tau(t,\varphi_s^{\mathbf{x}}(p))}(p)$$

by the Mean-Value Theorem, we get

$$\frac{1}{S}\sup_{\sigma\in(0,S]}\left|\int_0^\sigma f\circ h_t^\alpha\circ\varphi_s^{\mathbf{x}}(p)\,\mathrm{d}s\right|$$
$$\leq\|f\|_{\mathscr{C}^1}\cdot S+\frac{1}{S}\sup_{\sigma\in(0,S]}\left|\int_0^\sigma f\circ h_{e^s\,\tau(t,\varphi_s^{\mathbf{x}}(p))}(p)\,\mathrm{d}s\right|.$$

Mixing via shearing

Let $t \geq 1$ be fixed; we choose $S = t^{-\gamma/2}$ so that we can ignore the first term $\|f\|_{\mathscr{C}^1} \cdot S$ in the inequality above.
Denoting

$$p_s = \varphi_s^{\mathbf{X}}(p),$$

for every $\sigma \in (0, S]$ we need to bound

$$\left| \int_0^{\sigma} f \circ h_{e^s \tau(t, p_s)}(p) \, \mathrm{d}s \right|.$$

This is the integral of f along a horocycle orbit segment, but parametrized in a "complicated" way, which we need to understand.

The shear and the distortion

For our parametrization $s \mapsto e^s \tau(t, p_s)$, let us define

$$v(t,p,s) := \frac{\partial}{\partial s} e^s \tau(t, p_s) \qquad \text{the } \textbf{shear},$$

$$u(t,p,s) := -\frac{\partial}{\partial s} \frac{1}{v(t,p,s)} = \frac{\frac{\partial}{\partial s} v(t,p,s)}{v(t,p,s)^2} \qquad \text{the } \textbf{distortion}.$$

Proposition

There exists a constant $A \geq 1$ such that

$$\frac{t}{A} \leq |v(t,p,s)| \leq At,$$
$$0 \leq |u(t,p,s)| \leq At^{-1},$$

for all $p \in M$ and $s \in (0, S]$.

Conclusion of the proof

Let us finish the proof of mixing, assuming the previous proposition. For simplicity, we will assume that $t \geq 2A \geq 1$. We also assume that $f \in \mathscr{C}^4(M)$. We write

$$\begin{aligned} \int_0^\sigma f \circ h_{e^s\,\tau(t,p_s)}(p)\,\mathrm{d}s &= \int_0^\sigma f \circ h_{e^s\,\tau(t,p_s)}(p)\,\frac{v(t,p,s)}{v(t,p,s)}\,\mathrm{d}s \\ &= \frac{1}{v(t,p,\sigma)} \int_0^\sigma f \circ h_{e^s\,\tau(t,p_s)}(p)\,v(t,p,s)\,\mathrm{d}s \\ &\qquad + \int_0^\sigma u(t,p,s) \left(\int_0^s f \circ h_{e^l\,\tau(t,p_l)}(p)\,v(t,p,l)\,\mathrm{d}l \right) \mathrm{d}s, \end{aligned}$$

where we integrated by parts in the last equality.

Conclusion of the proof

Changing variables,

$$\int_0^{\sigma} f \circ h_{e^s\,\tau(t,p_s)}(p)\,\mathrm{d}s = \frac{1}{v(t,p,\sigma)} \int_{\tau(t,p)}^{e^{\sigma}\,\tau(t,p_\sigma)} f \circ h_s(p)\,\mathrm{d}s + \int_0^{\sigma} u(t,p,s) \left(\int_{\tau(t,p)}^{e^s\,\tau(t,p_s)} f \circ h_l(p)\,\mathrm{d}l \right) \mathrm{d}s.$$

Now we use the estimates on $|v(t,p,s)|$ and on $|u(t,p,s)|$ to obtain

$$\left| \int_0^{\sigma} f \circ h_{e^s\,\tau(t,p_s)}(p)\,\mathrm{d}s \right| \leq \frac{2A}{t} \sup_{s\in(0,\sigma]} \left| \int_0^{e^s\,\tau(t,p_s)-\tau(t,p)} f \circ h_l(h_t^{\alpha}(p))\,\mathrm{d}l \right|.$$

Conclusion of the proof

Note that, by the Mean-Value Theorem and the bounds on $v(p,t,s) = \frac{\partial}{\partial s} e^s \tau(t,p_s)$,

$$|e^s \tau(t,p_s) - \tau(t,p)| \leq Ast.$$

Therefore, up to increasing the constant A, we deduce

$$\left| \int_0^\sigma f \circ h_{e^s \tau(t,p_s)}(p) \,\mathrm{d}s \right| \leq A \|f\|_{\mathscr{C}^4} \sigma^{1-\gamma} t^{-\gamma}.$$

Recalling we set $S = t^{-\gamma/2}$, we conclude

$$\begin{aligned} \frac{1}{S} \sup_{\sigma \in (0,S]} \left| \int_0^\sigma f \circ h_t^\alpha \circ \varphi_s^x(p) \,\mathrm{d}s \right| &\leq \|f\|_{\mathscr{C}^1} \cdot S + A \|f\|_{\mathscr{C}^4} (St)^{-\gamma} \\ &\leq A \|f\|_{\mathscr{C}^4} t^{-\gamma/2}. \end{aligned}$$

Polynomial mixing

We have proved the following theorem.

Theorem

There exists $\gamma \in (0,1)$ such that the following holds. Let $(h_t^\alpha)_{t\in\mathbb{R}}$ be a time-change of the horocycle flow on M. There exists $A > 0$ such that, for all $f \in \mathscr{C}^4(M)$ and $g \in \mathscr{C}^1(M)$, and for all $t \geq A$, we have

$$\left| \int_M f \circ h_t^\alpha \cdot g \,\mathrm{d}\mathrm{vol}^\alpha - \int_M f \,\mathrm{d}\mathrm{vol}^\alpha \int_M g \,\mathrm{d}\mathrm{vol}^\alpha \right| \leq A \|f\|_{\mathscr{C}^4} \|g\|_{\mathscr{C}^1} t^{-\gamma}.$$

Mixing results for time-changes of horocycle flows have been proved by

- Kushnirenko (1974) and Marcus (1977) in non-quantitative form,
- Forni-Ulcigrai (2012) in the case we discussed here,
- R. (2022) for time-changes of unipotent flows on finite-volume (possibly non-compact) spaces.